Radiative Processes in Astrophysics

Lecture 9

Nov 13 (Wed.), 2013

(last updated Nov. 13)

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Equation of Motion

- Equation of motion of an electron in a uniform magnetic field:

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \longrightarrow \frac{d\mathbf{v}_{\parallel}}{dt} = 0$$

$$\frac{d\mathbf{v}_{\perp}}{dt} = \frac{-e}{\gamma m_e c}\mathbf{v}_{\perp} \times \mathbf{B} \rightarrow \frac{d^2\mathbf{v}_{\perp}}{dt^2} = -\omega_B^2 \mathbf{v}_{\perp} \quad \text{where } \omega_{\mathrm{B}} \equiv \frac{eB}{\gamma m_e c}$$

- Solution:

$$\mathbf{v}(t) = v_{\perp}\left(-\hat{\mathbf{x}}\sin\omega_B t + \hat{\mathbf{y}}\cos\omega_B t\right) + \hat{\mathbf{z}} v_{\parallel} \qquad \text{where } v_{\parallel} = v\cos\alpha,\ v_{\perp} = v\sin\alpha$$

$$\mathbf{r}(t) = \frac{v_{\perp}}{\omega_B}\left(\hat{\mathbf{x}}\cos\omega_B t + \hat{\mathbf{y}}\sin\omega_B t\right) + \hat{\mathbf{z}} v_{\parallel} t$$

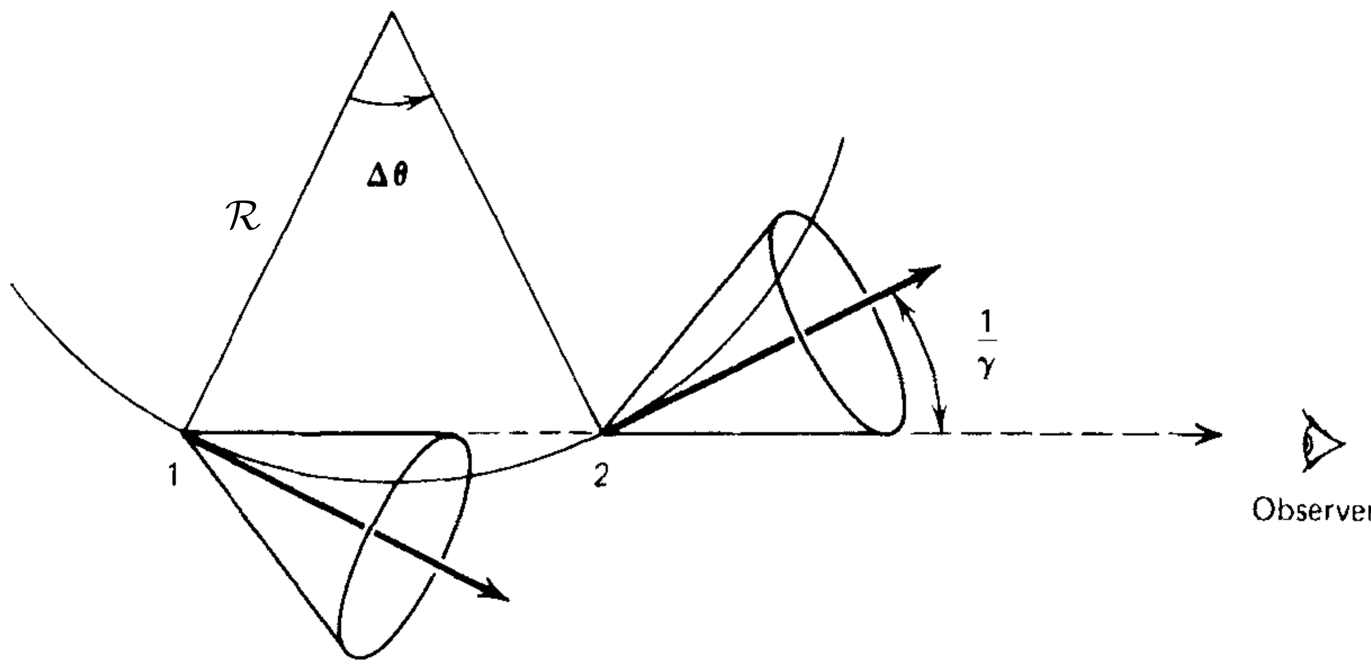
$$\omega_L = \frac{eB}{m_e c} \qquad \text{: Larmor frequency}$$

$$\omega_B = \frac{eB}{\gamma m_e c} = \frac{\omega_L}{\gamma}$$

$$= \frac{17.6}{\gamma}\left(\frac{B}{\mu\mathrm{G}}\right) \ (\mathrm{Hz})$$

$$r_B = \frac{v_{\perp}}{\omega_B} = 1.7 \times 10^9 \gamma\beta_{\perp}\left(\frac{B}{\mu\mathrm{G}}\right) \ (\mathrm{cm})$$

[Spectrum of Synchrotron Radiation: A Qualitative Discussion]



- Because of beaming effects the emitted radiation fields appear to be concentrated in a narrow set of directions about the particle's velocity.

  The observer will see a pulses of radiation confined to a time interval much smaller than the gyration period. The spectrum will thus be spread over a much broader frequency range than on of order ω_B.

  The cone of emission has an angular width $\sim 1/\gamma$. Therefore, the observer will sees emission over the angular range of $\Delta\theta \simeq 2/\gamma$.

- The radiation appears beamed toward the direction of the observer in a series of pulses spaced in time $2\pi/\omega_B$ apart, but with each pulse lasting only $\Delta\theta \simeq 2/\gamma$.

- To Fourier analyze the pulse shape, we need to calculate the interval of the arrival times of the pulse. Let's consider a instantaneous rest frame of the electron.

The path length from point 1 to 2 is $\Delta s = \mathcal{R}\Delta\theta$, where $\mathcal{R}$ is the radius of curvature of the path.

The equation of motion:

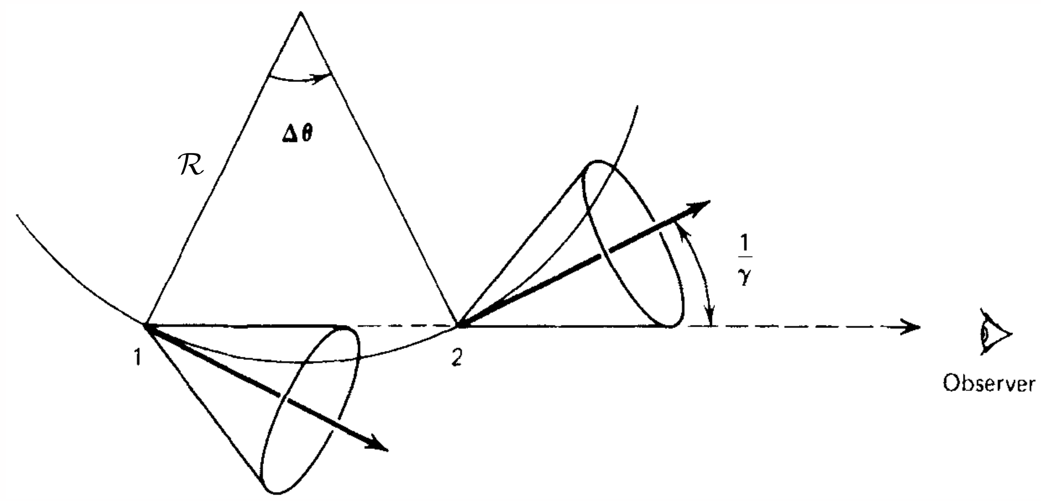
$$\gamma m_e \frac{\Delta \mathbf{v}}{\Delta t} = \frac{e}{c}\mathbf{v}\times\mathbf{B}$$

Since $|\Delta \mathbf{v}| = v\Delta\theta$ and $\Delta s = v\Delta t$, we find

$$\gamma m_e \frac{v\Delta\theta}{\Delta s / v} = \frac{e}{c} vB\sin\alpha \rightarrow \frac{\Delta\theta}{\Delta s} = \frac{eB\sin\alpha}{\gamma m_e c v} = \frac{\omega_B}{v}\sin\alpha \rightarrow \mathcal{R} = \frac{\Delta s}{\Delta\theta} = \frac{v}{\omega_B \sin\alpha}$$

Note that the curvature is different from the gyroradius. Therefore the path length is given by

$$\Delta s = 2\mathcal{R}/\gamma = \frac{2v}{\gamma\omega_B\sin\alpha} = \frac{2v}{\omega_L\sin\alpha}$$



Time interval that the particle passes from point 1 to 2:

$$\Delta t = t_2 - t_1 = \frac{\Delta s}{v} \simeq \frac{2}{\omega_L \sin\alpha}$$

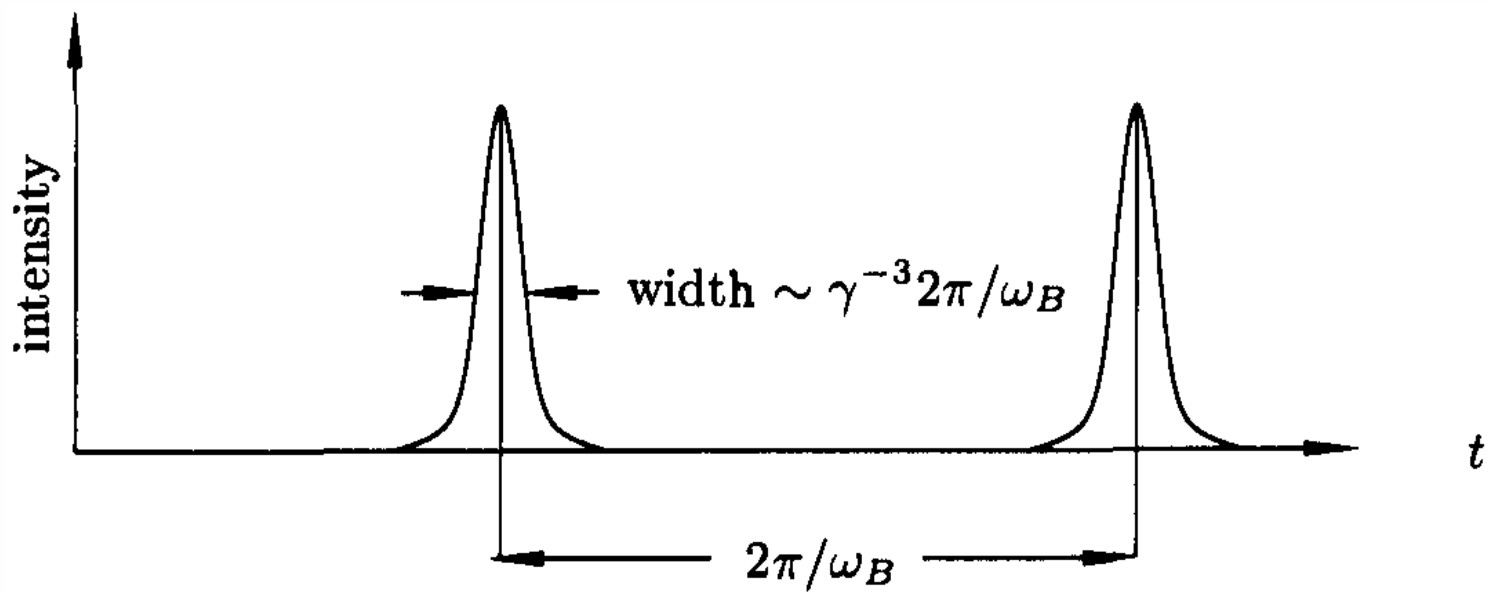
Note that point 2 is closer than point by $\Delta s / c$. The difference of the arrival times of the pulse is

$$\Delta t^A = t_2^A - t_1^A = \Delta t - \frac{\Delta s}{c} = \Delta t\left(1-\frac{v}{c}\right) = \frac{2}{\omega_L\sin\alpha}\left(1-\frac{v}{c}\right) \approx \frac{1}{\gamma^2\omega_L\sin\alpha} \qquad \leftarrow \qquad 1-\frac{v}{c}\approx\frac{1}{2\gamma^2}$$

$$\Delta t^A = t_2^A - t_1^A \approx \frac{1}{\gamma^2 \omega_L \sin\alpha} = \frac{1}{\gamma^3 \omega_B \sin\alpha}$$

Therefore, the width of the observed pulses is smaller than the gyration by a factor γ^3.

- Temporal pattern of received pulses:



- We define a critical frequency: $\omega_c \equiv \frac{3}{2}\gamma^2 \omega_L \sin\alpha = \frac{3}{2}\gamma^3 \omega_B \sin\alpha$

From the properties of Fourier transformation, we expect that the spectrum will be fairly broad, cutting off at frequencies like $1/\Delta t^A \approx \omega_c \approx \gamma^2 \omega_L = \gamma^3 \omega_B$.

- We can derive an important property of the spectrum for the synchrotron radiation.

  Remember that the electric field is a function of $\gamma\theta$, where θ is a polar angle about the direction of motion, because of the beaming effect. Then we can write

$$E(t) \propto F(\gamma\theta)$$

  Let time $t = 0$ and the path length $s = 0$ when the pulse is centered on the observer. Then, we find

$$\theta \approx s/\mathcal{R} \quad \text{and} \quad t \approx (s/v)(1 - v/c) \approx (s/v)/(2\gamma^2)$$

  Then we have

$$\gamma\theta \approx \gamma\frac{s}{\mathcal{R}} = \gamma\left(\frac{s}{v}\omega_B \sin\alpha\right) = \gamma\left(2\gamma^2 t\omega_B \sin\alpha\right) \propto \omega_c t$$

  The time dependence of the electric field can be written as

$$E(t) \propto g(\omega_c t)$$

  The Fourier transform of the electric field is

$$\begin{aligned}\hat{E}(\omega) &\propto \int_{-\infty}^{\infty} g\left(\omega_c t\right) e^{i\omega t}\, dt \quad \leftarrow \quad \xi \equiv \omega_c t \\ &= \int_{-\infty}^{\infty} g\left(\xi\right) e^{i(\omega/\omega_c)\xi}\, d\xi\end{aligned}$$

  Therefore, the power per unit frequency is a function of ω/ω_c: $\quad P(\omega) \propto \left|\hat{E}(\omega)\right|^2 = C_1 F\left(\frac{\omega}{\omega_c}\right)$

[Spectral Index for Power-law Electron Distribution]

- Often the number density of particles with energies between E and $E + dE$ can be approximagely expressed in the form:

$$N(\gamma)d\gamma = C\gamma^{-p}d\gamma \ \ (\gamma_1 < \gamma < \gamma_2) \ \text{ or } \ N(E)dE = CE^{-p}dE \ \ (E_1 < E < E_2)$$

- The total power radiated per unit volume per unit frequency by such a distribution is given by

$$\begin{aligned} P_{\text{tot}}(\omega) &= \int_{\gamma_1}^{\gamma_2} N(\gamma)P(\omega)d\gamma \\ &\propto \int_{\gamma_1}^{\gamma_2} \gamma^{-p} F\left(\frac{\omega}{\omega_c}\right) d\gamma \quad \leftarrow \quad x \equiv \omega / \omega_c \propto \gamma^{-2}\omega \\ &\propto \omega^{-(p-1)/2} \int_{x_1}^{x_2} x^{-(p-3)/2} F(x)\,dx \end{aligned}$$

- Then, the spectrum is also a power law and the power-law spectral index s is related to the particle distribution index p by

$$\begin{aligned} P_{\text{tot}}(\omega) &\propto \omega^{-s} \\ &\propto \omega^{-(p-1)/2} \end{aligned} \quad \longrightarrow \quad s = \frac{p-1}{2}$$

[Spectrum of Synchrotron Radiation: A Detailed Discussion]

- We will use the formula derived in Chapter 3 (Lecture 4).

$$\frac{dW}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int \mathbf{n}\times(\mathbf{n}\times\beta)\exp\left[i\omega\left(t'-\frac{\mathbf{n}\cdot\mathbf{r}(t')}{c}\right)\right]dt'\right|^2$$

- The coordinate system is chosen so that the particle has velocity **v** along the x' axis at time $t' = 0$.

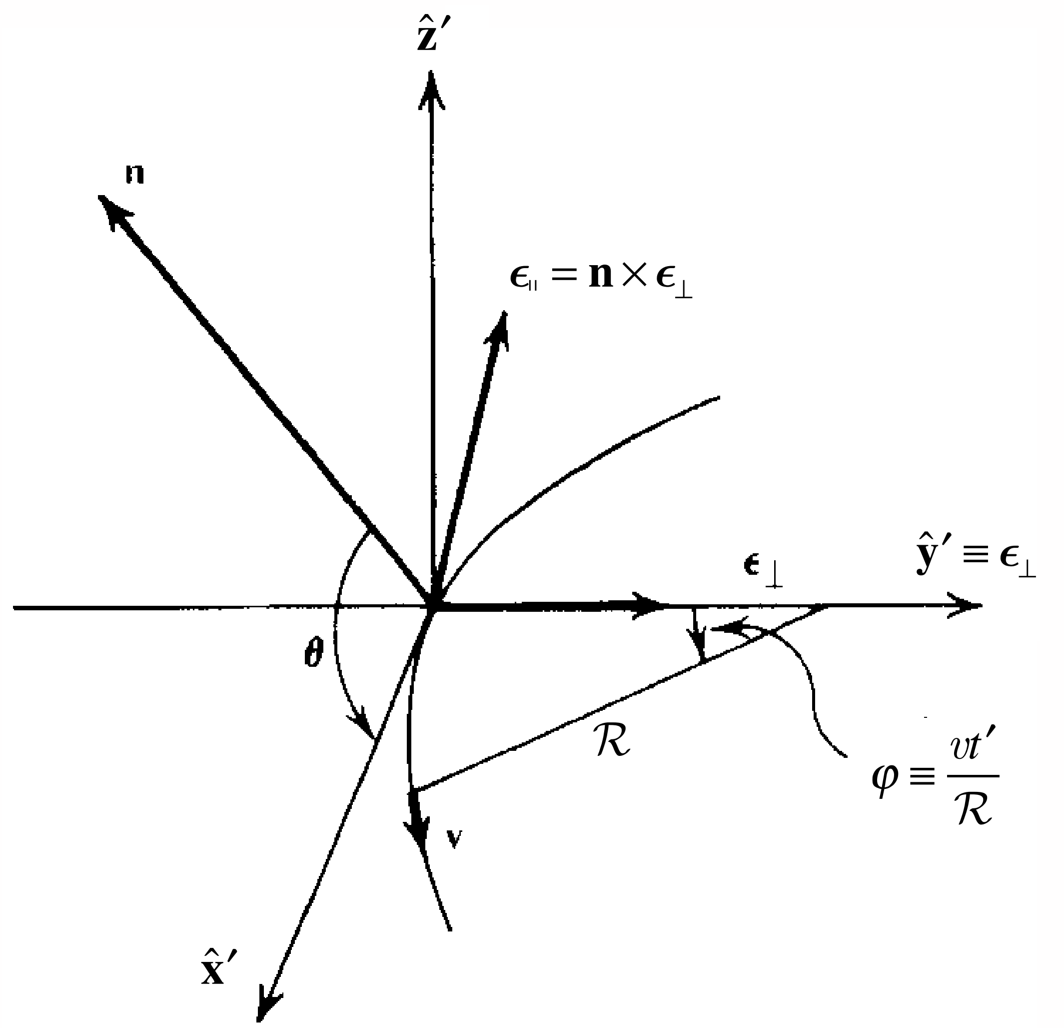
  $\epsilon_\perp$ is a unit vector along the y' axis in the orbital (x'-y') plane.

  Let θ represent the angle between the observing direction (**n**) and the velocity vector **v** at $t' = 0$.

  Then, an equivalent circular orbit at t' is given by

$$\mathbf{v}(t') = v\left(\hat{\mathbf{x}}'\cos\varphi + \hat{\mathbf{y}}'\sin\varphi\right), \text{ where } \varphi \equiv \frac{vt'}{\mathcal{R}} = (\omega_B \sin\alpha)t'$$

$$\mathbf{r}(t') = \mathcal{R}\left(\hat{\mathbf{x}}'\sin\varphi - \hat{\mathbf{y}}'\cos\varphi\right)$$



Note that (1) $\mathbf{n}\times\hat{\mathbf{x}}' = \sin\theta\,\hat{\mathbf{y}}'$, (2) $\mathbf{n}\times\hat{\mathbf{y}}' = \mathbf{n}\times\epsilon_\perp = \epsilon_\parallel$

(3) $\mathbf{n}\cdot\hat{\mathbf{y}}' = 0$, and (4) $\mathbf{n}\cdot\hat{\mathbf{x}}' = \cos\theta$

$$\begin{aligned}\mathbf{n}\times(\mathbf{n}\times\beta) &= \beta\left\{\mathbf{n}\times(\mathbf{n}\times\hat{\mathbf{x}}')\cos\varphi + \mathbf{n}\times(\mathbf{n}\times\hat{\mathbf{y}}')\sin\varphi\right\} \\ &= \beta\mathbf{n}\times(\mathbf{n}\times\hat{\mathbf{x}}')\cos\varphi + \beta\left(\mathbf{n}(\mathbf{n}\cdot\hat{\mathbf{y}}') - \hat{\mathbf{y}}'\right)\sin\varphi \\ &= \epsilon_\parallel\beta\sin\theta\cos\varphi - \epsilon_\perp\beta\sin\varphi\end{aligned}$$

- We note that

$$
\begin{aligned}
t' - \frac{\mathbf{n}\cdot\mathbf{r}(t')}{c} &= t' - \frac{\mathcal{R}}{c}\cos\theta\sin\varphi \;\leftarrow\; \mathbf{n}\cdot\hat{\mathbf{x}}' = \cos\theta \\
&\approx t' - \frac{\mathcal{R}}{c}\left(1-\frac{\theta^2}{2}\right)\left(\varphi - \frac{\varphi^3}{6}\right) \;\leftarrow\; \varphi = \frac{vt'}{\mathcal{R}} \\
&= t'\left[1-\frac{v}{c}\left(1-\frac{\theta^2}{2}\right)\left(1-\frac{(vt')^2}{6\mathcal{R}^2}\right)\right] \;\leftarrow\; 1-\frac{v}{c} \approx \frac{1}{2\gamma^2} \\
&\approx t'\left[1-\left(1-\frac{1}{2\gamma^2}\right)\left(1-\frac{\theta^2}{2}\right)\left(1-\left(1-\frac{1}{2\gamma^2}\right)^2\frac{c^2t'^2}{6\mathcal{R}^2}\right)\right] \\
&\approx \frac{t'}{2\gamma^2}\left[2\gamma^2-\left(2\gamma^2-1\right)\left(1-\frac{\theta^2}{2}\right)\left(1-\frac{c^2t'^2}{6\mathcal{R}^2}\right)\right] \leftarrow\; ct' \ll \mathcal{R}, \theta \ll 1 \\
&\approx \frac{t'}{2\gamma^2}\left[2\gamma^2-\left(2\gamma^2-1\right)\left(1-\frac{\theta^2}{2}-\frac{c^2t'^2}{6\mathcal{R}^2}\right)\right] \\
&\approx \frac{t'}{2\gamma^2}\left[2\gamma^2-\left(2\gamma^2-1\right)+2\gamma^2\left(\frac{\theta^2}{2}+\frac{c^2t'^2}{6\mathcal{R}^2}\right)\right] \\
&= \frac{1}{2\gamma^2}\left[\left(1+\gamma^2\theta^2\right)t' + \frac{c^2\gamma^2t'^3}{3\mathcal{R}^2}\right]
\end{aligned}
$$

- We also note that

$$\mathbf{n}\times(\mathbf{n}\times\beta) = \boldsymbol{\epsilon}_{\parallel}\beta\sin\theta\cos\varphi - \boldsymbol{\epsilon}_{\perp}\beta\sin\varphi \;\leftarrow\; \beta\approx 1$$
$$\approx \boldsymbol{\epsilon}_{\parallel}\sin\theta\cos\varphi - \boldsymbol{\epsilon}_{\perp}\sin\varphi$$
$$\approx -\boldsymbol{\epsilon}_{\perp}\varphi + \boldsymbol{\epsilon}_{\parallel}\theta = -\boldsymbol{\epsilon}_{\perp}\frac{vt'}{\mathcal{R}} + \boldsymbol{\epsilon}_{\parallel}\theta$$
$$\approx -\boldsymbol{\epsilon}_{\perp}\frac{ct'}{\mathcal{R}} + \boldsymbol{\epsilon}_{\parallel}\theta$$

$$t' - \frac{\mathbf{n}\cdot\mathbf{r}(t')}{c} \approx \frac{1}{2\gamma^2}\left[\left(1+\gamma^2\theta^2\right)t' + \frac{c^2\gamma^2 t'^3}{3\mathcal{R}^2}\right]$$

- We can identify the contribution to the received power in the two orthogonal polarized directions.

$$\frac{dW}{d\omega d\Omega} \equiv \frac{dW_{\parallel}}{d\omega d\Omega} + \frac{dW_{\perp}}{d\omega d\Omega} \longleftarrow \frac{dW}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int \mathbf{n}\times(\mathbf{n}\times\beta)\exp\left[i\omega\left(t' - \frac{\mathbf{n}\cdot\mathbf{r}(t')}{c}\right)\right]dt'\right|^2$$

$$\frac{dW_{\parallel}}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int \frac{ct'}{\mathcal{R}}\exp\left[\frac{i\omega}{2\gamma^2}\left(\theta_\gamma^2 t' + \frac{c^2\gamma^2 t'^3}{3\mathcal{R}^2}\right)\right]dt'\right|^2$$

$$\longleftarrow \theta_\gamma^2 \equiv 1+\gamma^2\theta^2$$

$$\frac{dW_{\perp}}{d\omega d\Omega} = \frac{e^2\omega^2\theta^2}{4\pi^2 c}\left|\int \exp\left[\frac{i\omega}{2\gamma^2}\left(\theta_\gamma^2 t' + \frac{c^2\gamma^2 t'^3}{3\mathcal{R}^2}\right)\right]dt'\right|^2$$

Define the following variables

$$y \equiv \gamma\frac{ct'}{\mathcal{R}\theta_\gamma}, \text{ and } \eta \equiv \frac{\omega\mathcal{R}\theta_\gamma^3}{3c\gamma^3} \longrightarrow$$

$$\frac{dW_{\parallel}}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left(\frac{\mathcal{R}\theta_\gamma^2}{\gamma^2 c}\right)^2\left|\int_{-\infty}^{\infty} y\exp\left[\frac{3}{2}i\eta\left(y+\frac{1}{3}y^3\right)\right]dt'\right|^2$$

$$\frac{dW_{\perp}}{d\omega d\Omega} = \frac{e^2\omega^2\theta^2}{4\pi^2 c}\left(\frac{\mathcal{R}\theta_\gamma}{\gamma c}\right)^2\left|\int \exp\left[\frac{3}{2}i\eta\left(y+\frac{1}{3}y^3\right)\right]dt'\right|^2$$

- The integrals are functions only of the parameter η. Since most of the radiation occurs at angle $\theta \approx 0$, η can be written as

$$\eta \approx \eta(\theta=0) = \frac{\omega\mathcal{R}}{3c\gamma^3} = \frac{\omega v}{3c\gamma^3\omega_B\sin\alpha} \approx \frac{\omega}{2\omega_c} \qquad \text{where } \omega_c \equiv \frac{3}{2}\frac{\gamma^2 eB\sin\alpha}{m_e c} = \frac{3}{2}\gamma^3\omega_B\sin\alpha$$

The frequency dependence of the spectrum depends on ω only through ω/ω_c.
The angular dependence uses θ only through the combination $\gamma\theta$.

- The integrals can be expressed in terms of the modified Bessel functions of 1/3 and 2/3 order.

From 10.4.26, 10.4.31, and 10.4.32 of Abramovitz & Stegun (1965)
See Westfold 1959, ApJ, 130, 241

$$\frac{dW_{\parallel}}{d\omega d\Omega} = \frac{e^2\omega^2}{3\pi^2 c}\left(\frac{\mathcal{R}\theta_\gamma^2}{\gamma^2 c}\right)^2 K_{2/3}^2(\eta)$$

$$\frac{dW_{\perp}}{d\omega d\Omega} = \frac{e^2\omega^2\theta^2}{3\pi^2 c}\left(\frac{\mathcal{R}\theta_\gamma}{\gamma c}\right)^2 K_{1/3}^2(\eta)$$

- The energy per frequency range radiated by the particle per complete orbit in the projected normal plane can be obtained by integrating over solid angle.

- We note that the emitted radiation is almost completely confined to the solid angle shown shaded in the following figure, which lies within an angle $1/\gamma$ of a cone of half-angle α. Therefore, the integral over the solid angle can be approximated by

$$\frac{dW_{\parallel}}{d\omega} = \int_0^{\pi} \frac{dW_{\parallel}}{d\omega d\Omega} 2\pi \sin\theta \, d\theta \approx \int_{-\infty}^{\infty} \frac{dW_{\parallel}}{d\omega d\Omega} 2\pi \sin\alpha \, d\theta$$

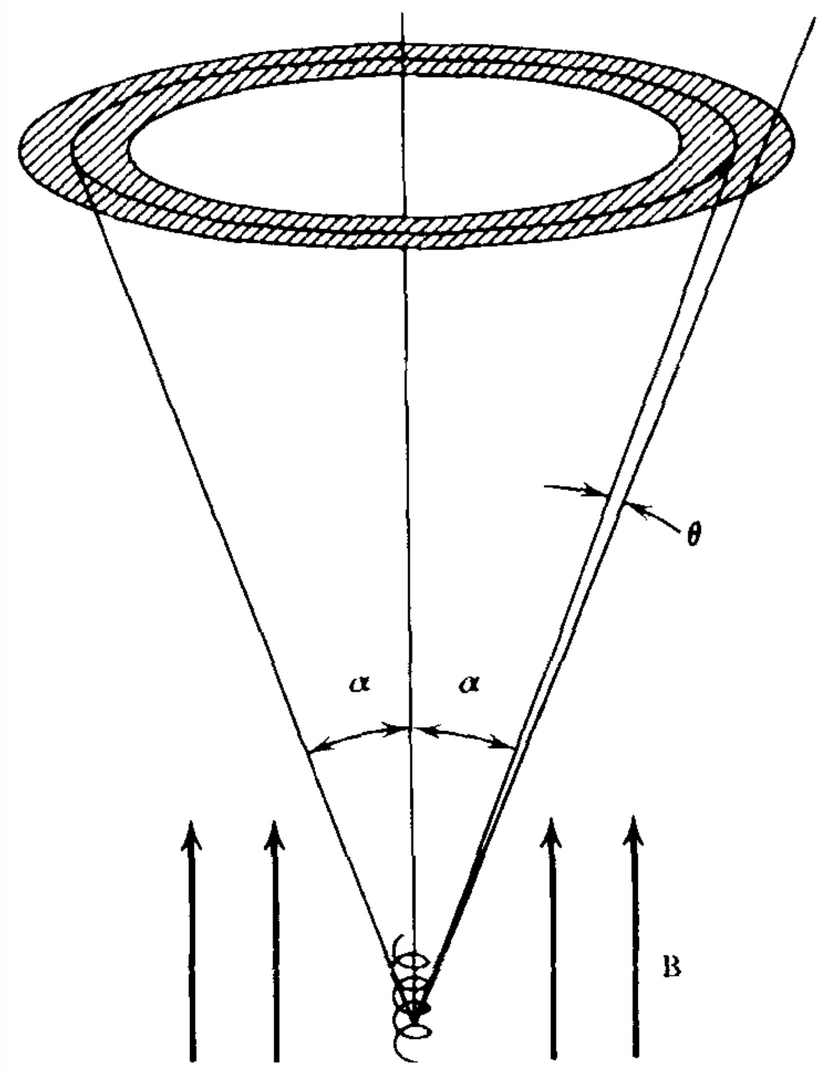
- Therefore,

$$\frac{dW_{\parallel}}{d\omega} = \frac{2e^2\omega^2\mathcal{R}^2 \sin\alpha}{3\pi c^3\gamma^4} \int_{-\infty}^{\infty} \theta_\gamma^4 K_{2/3}^2(\eta) \, d\theta$$

$$\frac{dW_{\perp}}{d\omega} = \frac{2e^2\omega^2\mathcal{R}^2 \sin\alpha}{3\pi c^3\gamma^2} \int_{-\infty}^{\infty} \theta_\gamma^2 K_{1/3}^2(\eta) \, d\theta$$

- The emitted power per frequency is obtained by dividing the orbital period of the charge $T = 2\pi / \omega_B$:

$$P_{\parallel}(\omega) \equiv \frac{1}{T} \frac{dW_{\parallel}}{d\omega}, \quad P_{\perp}(\omega) \equiv \frac{1}{T} \frac{dW_{\perp}}{d\omega}$$



- Emitted power:

$$P_\perp(\omega) \equiv \frac{\sqrt{3}e^3 B\sin\alpha}{4\pi m_e c^2}\left[F(x)+G(x)\right]$$

$$P_\parallel(\omega) \equiv \frac{\sqrt{3}e^3 B\sin\alpha}{4\pi m_e c^2}\left[F(x)-G(x)\right]$$

where

$$F(x) \equiv x\int_x^\infty K_{5/3}(\xi)\,d\xi$$

$$G(x) \equiv xK_{2/3}(x)$$

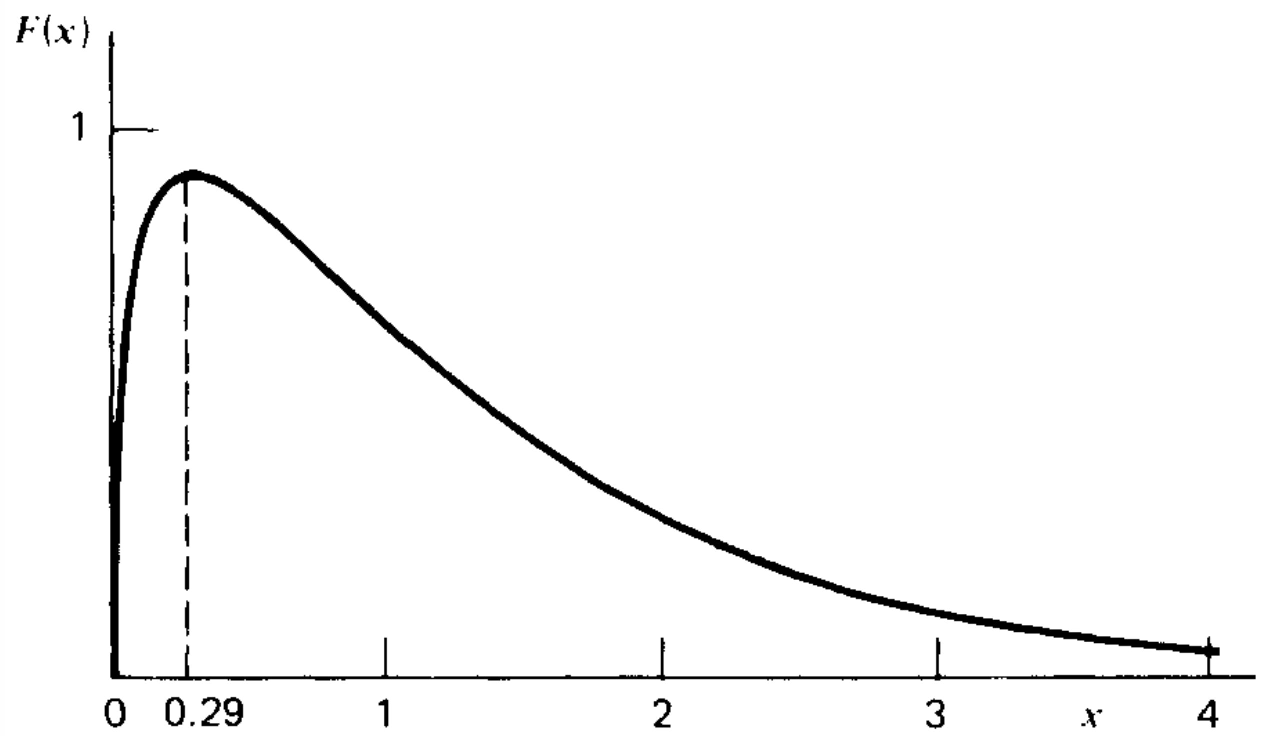
$$x \equiv \omega/\omega_c$$

- Total emitted power per frequency:

$$P(\omega) \equiv P_\parallel(\omega)+P_\perp(\omega) = \frac{\sqrt{3}e^3 B\sin\alpha}{2\pi m_e c^2}F(x)$$

$$F(x) \sim \frac{4\pi}{\sqrt{3}\Gamma(1/3)}\left(\frac{x}{2}\right)^{1/3}, \quad x \ll 1$$

$$F(x) \sim \left(\frac{\pi}{2}\right)^{1/2} e^{-x}x^{1/2}, \qquad x \gg 1$$



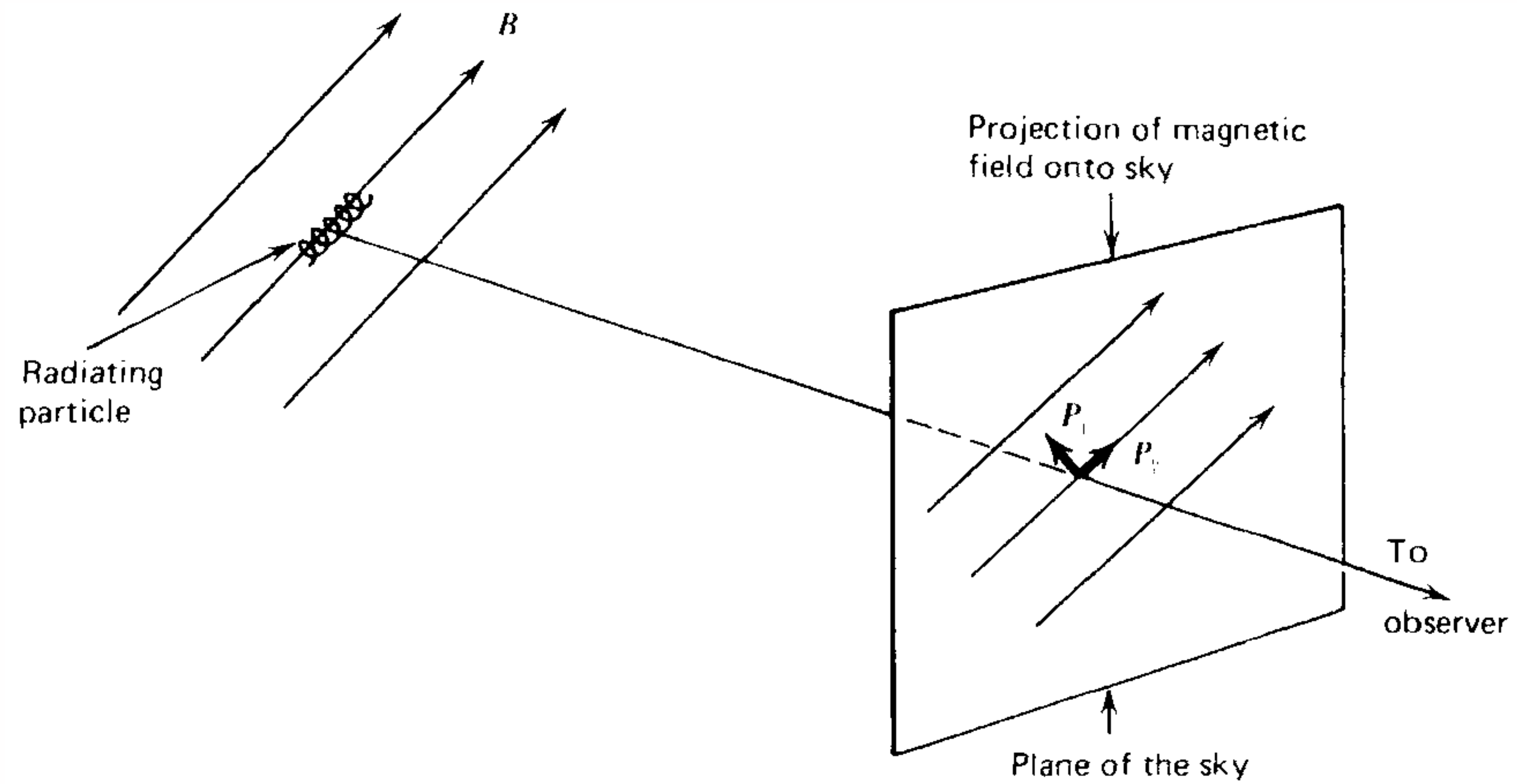
- For a power-law distribution of electrons $N(\gamma)d\gamma = C\gamma^{-p}d\gamma \ \ (\gamma_1 < \gamma < \gamma_2)$, we obtain the total power per unit volume per unit frequency:

$$
\begin{aligned}
P_{\text{tot}} &= \int N(\gamma)P(\omega)d\gamma \\
&\equiv \frac{\sqrt{3}e^3 CB\sin\alpha}{2\pi m_e c^2 (p+1)}\Gamma\left(\frac{p}{4}+\frac{19}{12}\right)\Gamma\left(\frac{p}{4}-\frac{1}{12}\right)\left(\frac{m_e c\omega}{3eB\sin\alpha}\right)^{-(p-1)/2} \\
&\propto \omega^{-(p-1)/2}
\end{aligned}
$$

- For the complete derivation of the formula, see Westfold (1959).

[Polarization of Synchrotron Radiation]

- In general, the radiation from a single charge will be elliptically polarized. Fro any reasonable distribution of particles that varies smoothly with pitch angle, the radiation will be partially linearly polarized.



- Degree of linear polarization of a single energy:

$$\Pi(\omega) \equiv \frac{P_\perp(\omega) - P_\parallel(\omega)}{P_\perp(\omega) + P_\parallel(\omega)} = \frac{G(x)}{F(x)}$$

- For particles with a power law distribution of energies:

$$\Pi(\omega) = \frac{\int G(x)\gamma^{-p}\, d\gamma}{\int F(x)\gamma^{-p}\, d\gamma} \leftarrow \gamma \propto x^{-1/2}$$

$$= \frac{\int G(x) x^{(p-3)/2}\, dx}{\int F(x) x^{(p-3)/2}\, dx} \longleftarrow$$

$$\int_0^\infty x^\mu F(x)\,dx = \frac{2^{\mu+1}}{\mu+2}\Gamma\left(\frac{\mu}{2}+\frac{7}{3}\right)\Gamma\left(\frac{\mu}{2}+\frac{2}{3}\right)$$

$$\int_0^\infty x^\mu G(x)\,dx = 2^\mu\Gamma\left(\frac{\mu}{2}+\frac{4}{3}\right)\Gamma\left(\frac{\mu}{2}+\frac{2}{3}\right)$$

$$= \frac{(p+1)/2}{2}\frac{1}{\frac{p-3}{4}+\frac{4}{3}}$$

$$= \frac{p+1}{p+\frac{7}{3}}$$

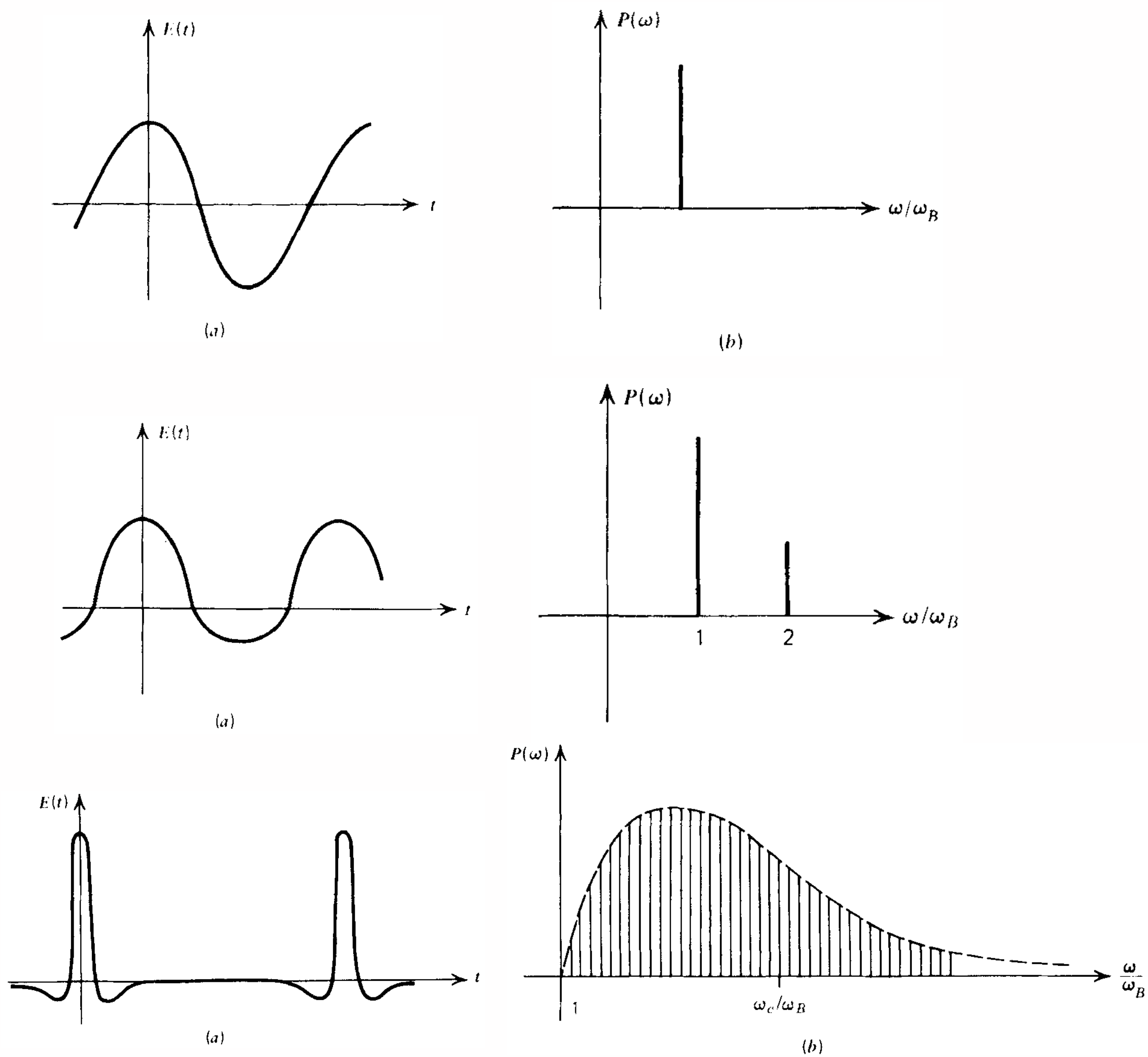
- For particles of a single energy, the polarization degree of the frequency integrated radiation is

$$\Pi = \frac{\int G(x)\,dx}{\int F(x)\,dx} = \frac{p+1}{p+\frac{7}{3}} \quad \leftarrow \quad p = 3$$

$$= \frac{3}{4} = 75\%$$

[Transition from Cyclotron to Synchrotron Emission]

- For low energies, the electric field components vary sinusoidally with the same frequency as the gyration in the magnetic field. The spectrum consists of a single line.
- When v/c increases, higher harmonics of the fundamental frequency begin to contribute.
- For very relativistic velocities, the originally sinusoidal form of *E(t)* has now become a series of sharp pulses, which is repeated at time intervals $2\pi/\omega_B$. The spectrum now involves a great number of harmonics, the envelope of which approaches the form of the function *F(x)*.



[Distinction between Received and Emitted Power]

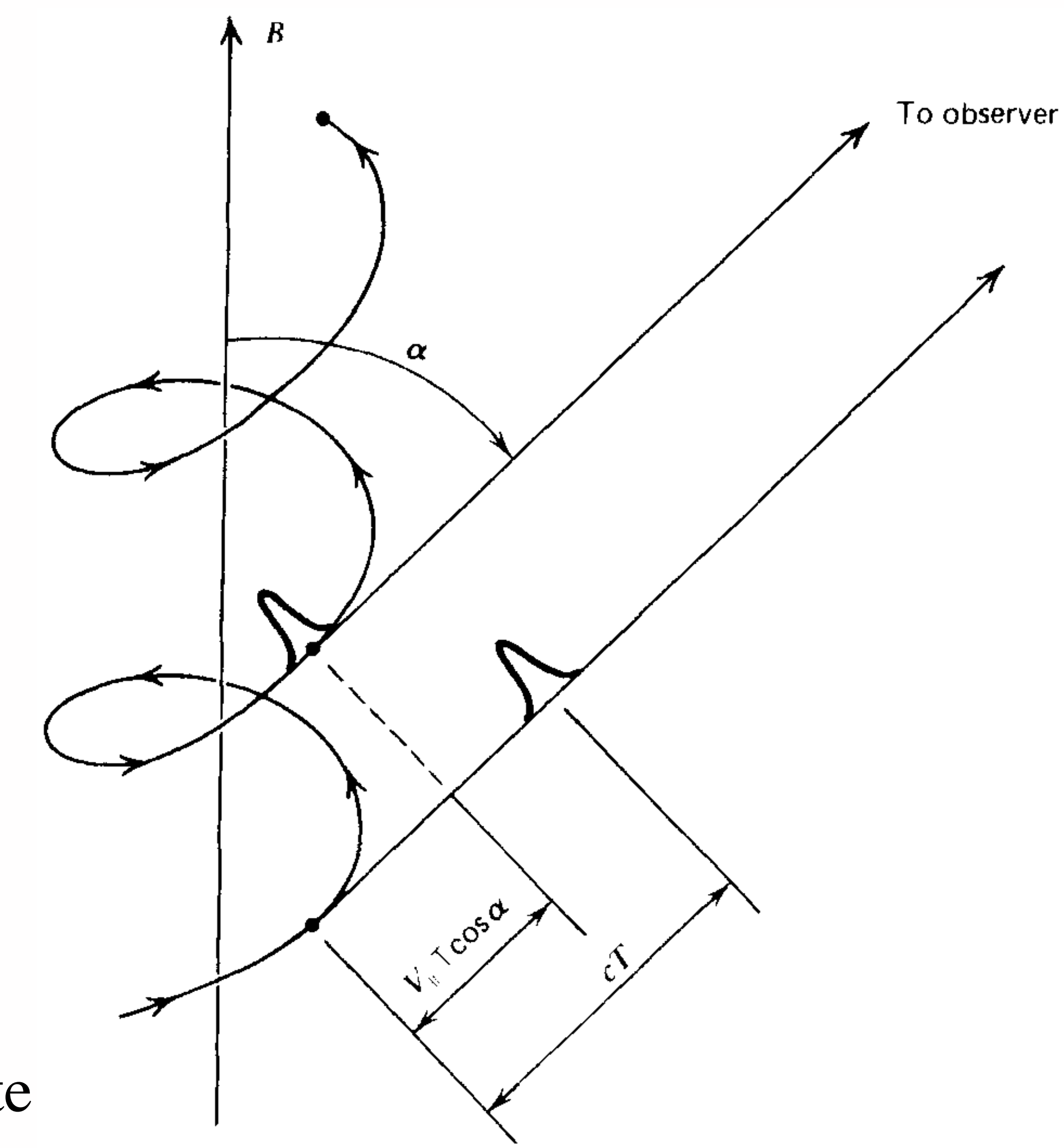
- If $T = 2\pi / \omega_B$ is the orbital period of the projected motion, then time-delay effects will give a period between the arrival of pulses T_A satisfying

$$T_A = T\left(1 - \frac{v_{\parallel}}{c}\cos\alpha\right) = T\left(1 - \frac{v}{c}\cos^2\alpha\right)$$

$$\approx T\left(1 - \cos^2\alpha\right) = \frac{2\pi}{\omega_B}\sin^2\alpha$$

Therefore, the fundamental observed frequency is

$\omega_B / \sin^2\alpha$ rather than ω_B.



- Two modifications to the preceding results:

(1) Spacing of the harmonics is $\omega_B / \sin^2\alpha$. For extreme relativistic particles this is not important, because one sees a continuum rather than the harmonic structure. Note that we did take the Doppler effects in deriving the pulse width Δt_A and consequently for the critical frequency ω_c. The continuum radiation is still a function of ω / ω_c.

(2) The emitted power was found by dividing the energy by the period T. But the received power must be obtained by dividing by T_A. Thus we have $P_r = P_e / \sin^2\alpha$.

However, these corrections are not important for most cases of interest.

[Synchrotron Self-Absorption]

- **Opacity**

We first need to generalize the Einstein coefficients to include continuum states.

For a given energy of a photon $h\nu$ there are many possible transitions, meaning that the absorption coefficient should be obtained by summing over all upper states 2 and lower stats 1:

$$\alpha_\nu = \frac{h\nu}{4\pi}\sum_{E_1}\sum_{E_2}\left[n(E_1)B_{12} - n(E_2)B_{21}\right]\phi_{21}(\nu)$$

The profile function $\phi_{21}(\nu)$ is essentially a Dirac delta-function: $\phi_{21}(\nu) = \delta\left(\nu - \frac{E_2 - E_1}{h}\right)$

In terms of the Einstein coefficients, the emitted power is given by

$$P(\nu, E_2) = h\nu\sum_{E_1} A_{21}\phi_{21}(\nu) \longleftarrow \qquad A_{21} = \left(2h\nu^3/c^2\right)B_{21}$$

$$= \left(2h\nu^3/c^2\right)h\nu\sum_{E_1} B_{21}\phi_{21}(\nu)$$

Absorption coefficient due to stimulated emission:

$$-\frac{h\nu}{4\pi}\sum_{E_1}\sum_{E_2} n(E_2)B_{21}\phi_{21}(\nu) = -\frac{c^2}{8\pi h\nu^3}\sum_{E_2} n(E_2)P(\nu, E_2)$$

True absorption coefficient:

$$\frac{h\nu}{4\pi}\sum_{E_1}\sum_{E_2} n(E_1)B_{12}\phi_{21}(\nu) = \frac{c^2}{8\pi h\nu^3}\sum_{E_2} n(E_2 - h\nu)P(\nu, E_2) \longleftarrow \qquad B_{12} = B_{21}$$

Therefore, we have

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3}\sum_{E_2}\left[n(E_2 - h\nu) - n(E_2)\right]P(\nu, E_2)$$

Let $f(p)d^3p \equiv$ number of electrons per volume with momenta in d^3p about p.

Number density of quantum states = $g\dfrac{d^3p}{h^3}$ ($g = 2$ for spin 1/2 particles)

Electron density per quantum state = $\dfrac{h^3}{g}f(p)$

Therefore, we can make the replacements

$$\sum\nolimits_2 \rightarrow \frac{g}{h^3}\int d^3p, \quad n(E_2) \rightarrow \frac{h^3}{g}f(p)$$

Then, the absorption coefficient becomes

$$\boxed{\alpha_\nu = \frac{c^2}{8\pi h\nu^3}\int d^3p_2\left[f(p^*) - f(p_2)\right]P(\nu, E_2)}$$

where p^* is the momentum corresponding to energy $E_2 - h\nu$.

- For a thermal distribution of particles

$$f(p) = K \exp\left[-\frac{E(p)}{kT}\right]$$

$$f(p^*) - f(p_2) = K \exp\left[-\frac{E_2 - h\nu}{kT}\right] - K \exp\left[-\frac{E_2}{kT}\right]$$
$$= f(p_2)\left(e^{h\nu/kT} - 1\right)$$

Thus, the absorption coefficient is

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3}\left(e^{h\nu/kT} - 1\right)\int d^3p_2 f(p_2) P(\nu, E_2)$$
$$= \frac{1}{4\pi}\frac{c^2}{2h\nu^3}\left(e^{h\nu/kT} - 1\right) 4\pi j_\nu$$

Therefore, we obtained the Kirchhoff's Law for thermal emission.

$$\alpha_\nu = \frac{j_\nu}{B_\nu(T)}$$

- For an isotropic, and extremely relativistic electron distribution:

$$E = pc, \quad N(E)dE = f(p)4\pi p^2 dp, \quad d^3 p = 4\pi p^2 dp$$

Then $$\alpha_\nu = \frac{c^2}{8\pi h\nu^3}\int dE P(\nu,E)E^2\left[\frac{N(E-h\nu)}{(E-h\nu)^2} - \frac{N(E)}{E^2}\right]$$

Assume that $h\nu \ll E$ (in fact, a necessary condition for the application of classical electrodynamics) and expand to first order in $h\nu$.

$$\alpha_\nu = -\frac{c^2}{8\pi\nu^2}\int dE P(\nu,E)E^2 \frac{\partial}{\partial E}\left[\frac{N(E)}{E^2}\right]$$

- For a power law distribution of particles:

$$-\frac{d}{dE}\left[\frac{N(E)}{E^2}\right] = (p+2)CE^{-(p+1)} = \frac{(p+2)N(E)}{E}$$

$$\begin{aligned}
\alpha_\nu &= \frac{(p+2)c^2}{8\pi\nu^2}\int dE P(\nu,E)\frac{N(E)}{E} \\
&\propto \nu^{-2}\int dE F(x)\frac{E^{-p}}{E} \quad \leftarrow \quad x \equiv \frac{\omega}{\omega_c} \propto \nu\gamma^{-2} \propto \nu E^{-2} \\
&\propto \nu^{-2}\int \nu^{1/2} x^{-3/2}\, dx F(x)\nu^{-(p+1)/2} x^{(p+1)/2} \\
&\propto \nu^{-(p+4)/2}
\end{aligned}$$

Note $\alpha_\nu \propto \nu^{-(p+4)/2}$ indicates that the synchrotron emission is optically thick at low frequencies and optically thin at high frequencies.

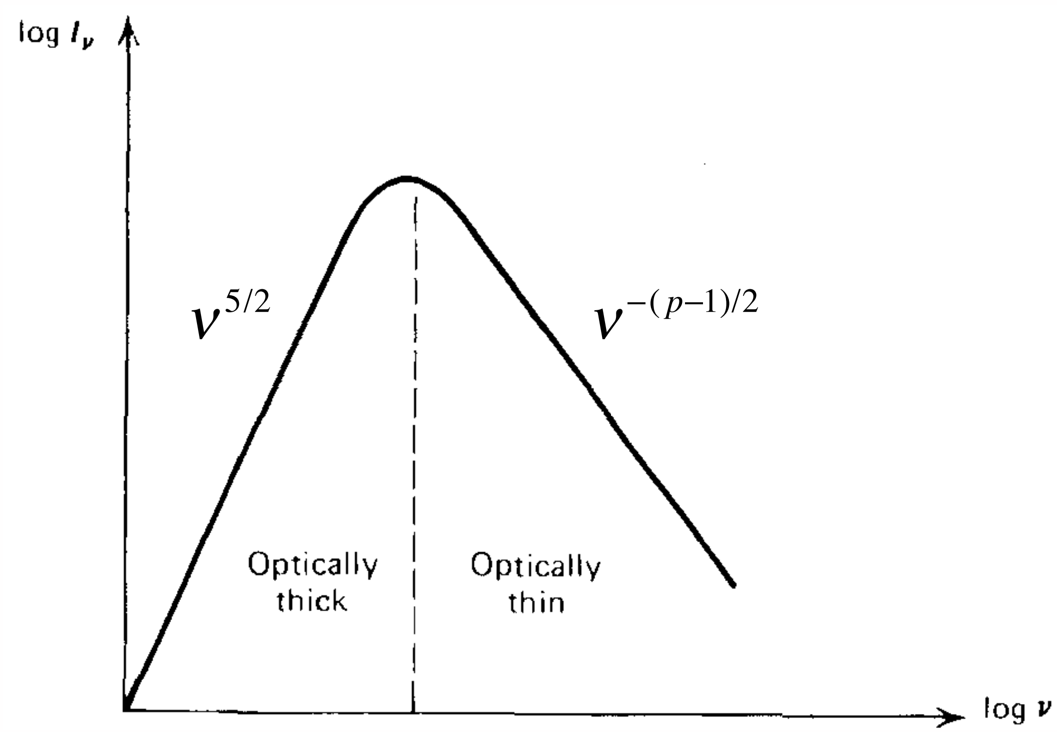
The source function is

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{P(\nu)}{4\pi\alpha_\nu} \propto \nu^{5/2}$$

For optically thin synchrotron emission, $I_\nu \propto j_\nu$

For optically thick emission, $I_\nu \propto S_\nu$

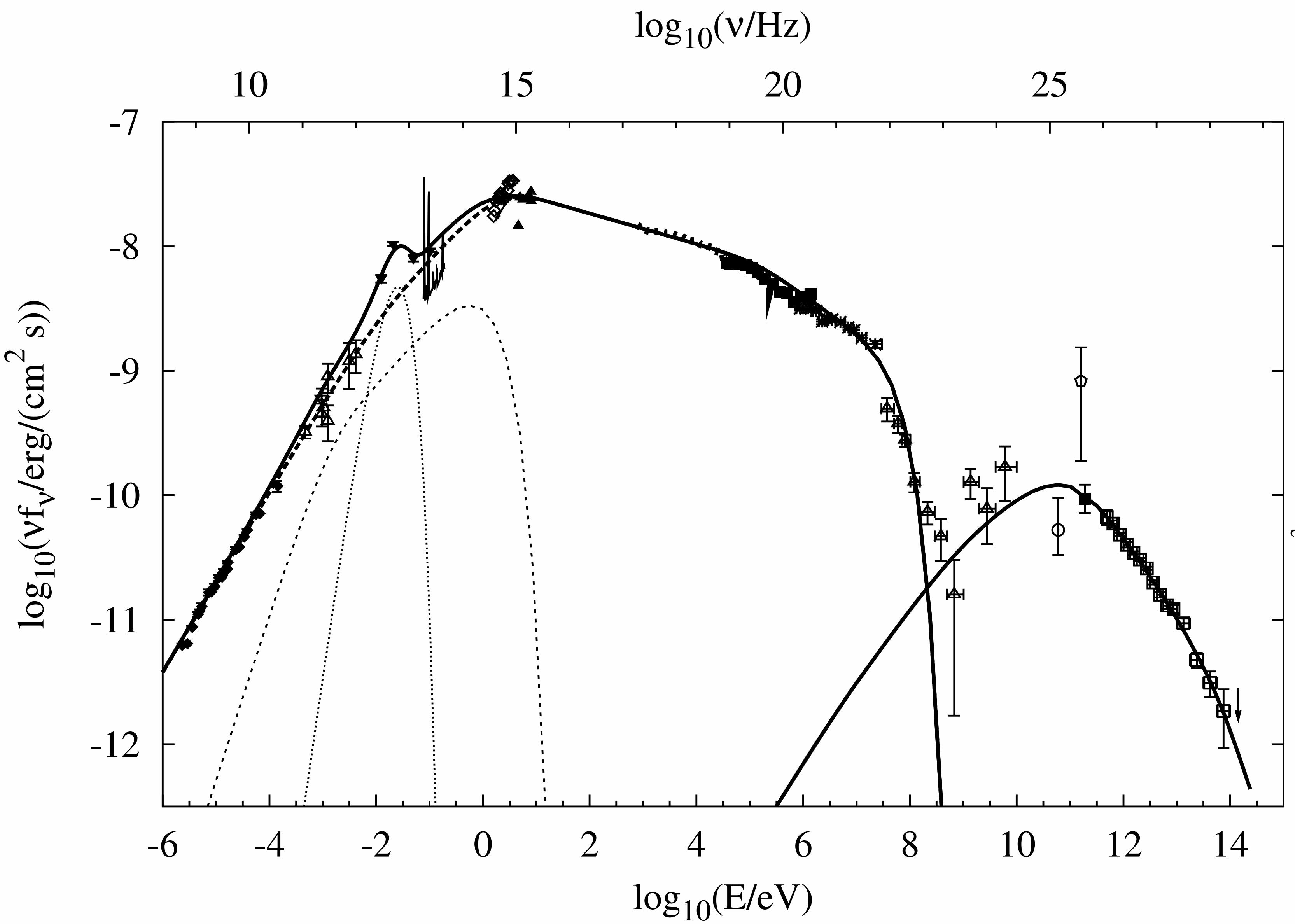
Therefore, the synchrotron spectrum from a power-law distribution of electrons is

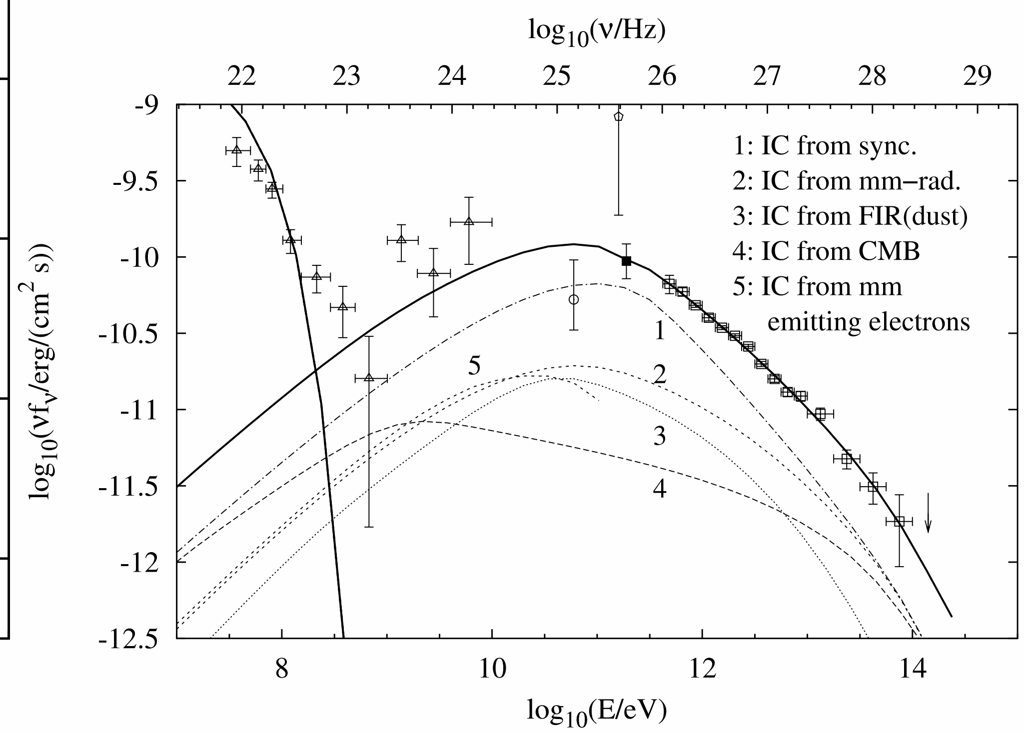


Astrophysical Example

- Crab nebula

Dots: modified blackbody with $T = 46$ K.
Thin dashed line: emission at mm wavelengths
Thick dashed line: synchrotron emission





Aharonian et al. (2004)

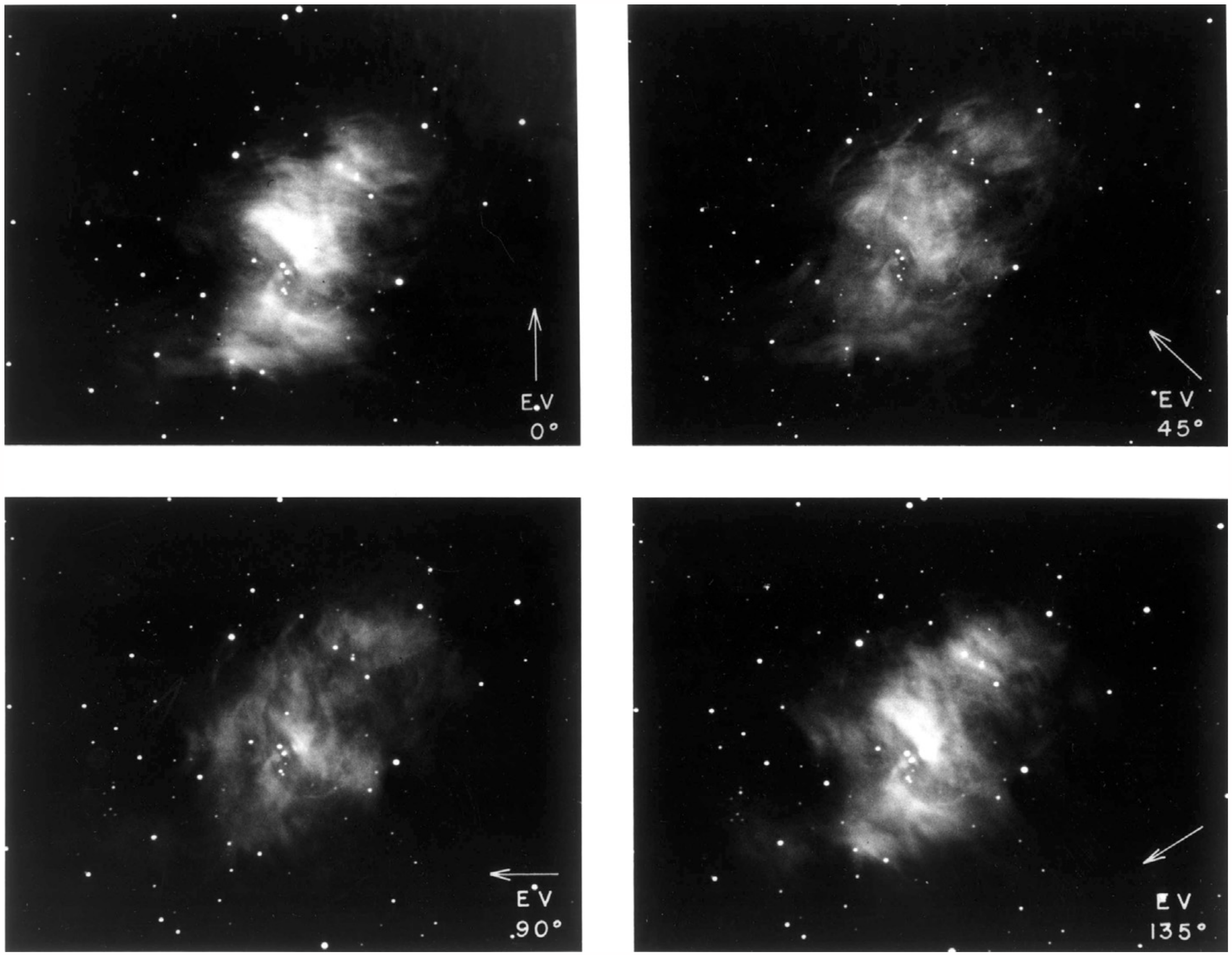


Fig. 8.3: Photographs of the Crab nebula in polarized light with the polarizer at different orientations. The arrows show the directions or planes of the transmitted transverse electric vector. Note the changing brightness pattern from photo to photo. The nebula has angular size $4' \times 6'$ and is $\sim$6 000 LY distant from the solar system. North is up and east to the left. The pulsar is the southwest (lower right) partner of the doublet at the center of the nebula. [Palomar Observatory/CalTech]