Radiative Processes in Astrophysics

Lecture 7
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Mathematical Formulae

- Gamma function

$$\Gamma(x) \equiv \int_0^\infty t^{x-1} e^{-t}\, dt, \quad \Gamma(x) = (x-1)! = (x-1)\Gamma(x-2), \quad \Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

- Euler-Mascheroni constant

$$\gamma \equiv \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right) = -\int_0^\infty e^{-x}\ln x\, dx = 0.577215664901532$$

- Modified Bessel function of the second kind

$$K_n(x) \equiv \frac{\Gamma(n+1/2)(2x)^n}{\sqrt{\pi}}\int_0^\infty \frac{\cos t}{\left(t^2+x^2\right)^{n+1/2}}dt$$

(1) $0 < x < \sqrt{n+1}$

$$K_n(x) \approx \begin{cases} -\ln(x/2) - \gamma & \text{if } n = 0 \\ \dfrac{\Gamma(n)}{2}\left(\dfrac{2}{x}\right)^n & \text{if } n > 0 \end{cases}$$

(2) $x \gg \left|n^2 - 1/4\right|$

$$K_n(x) \approx \sqrt{\frac{\pi}{2x}}e^{-x}\left[1 + \frac{(4n^2-1)}{8x}\right]$$

Recurrence formulae

$$K_{n-1}(x) - K_{n+1}(x) = -\frac{2n}{x}K_n(x)$$

$$K_{n-1}(x) + K_{n+1}(x) = -2K_n'(x)$$

Integral formula

$$\int xK_n^{\,2}(x)\,dx = \frac{1}{2}x^2\left[K_n^{\,2}(x) - K_{n-1}(x)K_{n+1}(x)\right]$$

$$= -xK_{n-1}(x)K_n(x) + \frac{1}{2}x^2\left[K_n^{\,2}(x) - K_{n-1}^{\,2}(x)\right]$$

[Covariance of Electromagnetic Phenomena]

- Equation of charge conservation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{c\,\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(-\frac{\partial}{c\,\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

The above equation can be written as a tensor equation,

$$\frac{\partial j^\mu}{\partial x^\mu} = 0, \quad j^\mu{}_{,\mu} = 0 \quad \text{or} \quad \partial_\mu j^\mu = 0$$

if the **four-current** is defined by

$$j^\mu = \begin{pmatrix} \rho c \\ \mathbf{j} \end{pmatrix} \qquad j_\mu = \begin{pmatrix} -\rho c \\ \mathbf{j} \end{pmatrix}$$

- Note that the Jacobian (determinant) of the transformation from x_μ to x'_μ is simply the determinant of Λ, which is unity. Therefore, the **four-volume element** is an invariant.

$$dx'_0 dx'_1 dx'_2 dx'_3 = \det \Lambda \; dx_0 dx_1 dx_2 dx_3 = dx_0 dx_1 dx_2 dx_3$$

Since ρ is the zeroth component of the four-current, the charge element within a three-volume element is an invariant.

$$de = \rho dx_1 dx_2 dx_3$$

It is also an empirical fact that e is invariant.

- The set of vector and scalar wave equations in the Lorentz gauge is

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j}$$

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{4\pi}{c}(\rho c)$$

If we define the **four-potential**

$$A^\mu = \begin{pmatrix} \phi \\ \mathbf{A} \end{pmatrix}, \quad A_\mu = \begin{pmatrix} -\phi \\ \mathbf{A} \end{pmatrix}$$

then the wave equations can be written as the tensor equations

$$\frac{\partial^2 A^\mu}{\partial x^\nu \partial x_\nu} = -\frac{4\pi}{c} j^\mu, \quad \partial_\nu \partial^\nu A^\mu = -\frac{4\pi}{c} j^\mu \quad \text{or} \quad A^{\mu,\nu}_{,\nu} = -\frac{4\pi}{c} j^\mu$$

d'Alembertian operator: $\Box \equiv \frac{\partial^2}{\partial x^\nu \partial x_\nu} \rightarrow \Box A^\mu = -\frac{4\pi}{c} j^\mu$

- The Lorentz gauge should be preserved under Lorentz transformations.

$$\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0 \rightarrow \frac{\partial A^\mu}{\partial x^\mu} = 0 \quad \text{or} \quad A^\mu{}_{,\mu} = 0$$

- **Electromagnetic field tensor**:

The fields are expressed in terms of the potentials as

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \nabla \times \mathbf{A}$$

The x components of the electric and magnetic fields are explicitly

$$E_x = -\frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial \phi}{\partial x} = \left(\partial^0 A^1 - \partial^1 A^0\right)$$
$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \left(\partial^2 A^3 - \partial^3 A^2\right)$$

These equations imply that the electric and magnetic fields, six components in all, are the elements of a **second-rank, antisymmetric field-strength tensor**, because a rank two antisymmetric tensor has exactly six independent components.

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \longrightarrow \begin{aligned} F^{0i} &= E_i \\ F^{i0} &= -E_i \\ F^{12} &= -F^{21} = B_3, \cdots \end{aligned}$$

covariant field-strength tensor

$$F_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}F^{\alpha\beta} \longrightarrow \begin{aligned} F_{0i} &= \eta_{0\alpha}\eta_{i\beta}F^{\alpha\beta} = -F^{0i} \\ F_{i0} &= \eta_{i\alpha}\eta_{0\beta}F^{\alpha\beta} = -F^{i0} \\ F_{ij} &= \eta_{i\alpha}\eta_{j\beta}F^{\alpha\beta} = F^{ij} \end{aligned}$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \longrightarrow \begin{aligned} F_{0i} &= -E_i \\ F_{i0} &= E_i \\ F_{12} &= -F_{21} = B_3, \cdots \end{aligned}$$

- The two Maxwell equations containing sources (inhomogeneous equations):

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 4\pi\rho \\ \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi}{c}\mathbf{j} \end{aligned} \longrightarrow \begin{aligned} \sum_{i=1}^{3}\partial_i E_i &= \frac{4\pi}{c} j^0 \\ \partial_2 B_3 - \partial_3 B_2 - \partial_0 E_1 &= \frac{4\pi}{c} j^1 \end{aligned} \longrightarrow \begin{aligned} -\sum_{i=1}^{3}\partial_i F^{i0} &= \frac{4\pi}{c} j^0 \\ -\partial_0 F^{01} - \partial_2 F^{21} - \partial_3 F^{31} &= \frac{4\pi}{c} j^1 \end{aligned}$$

$$\boxed{\partial_\mu F^{\mu\nu} = -\frac{4\pi}{c} j^\nu} \quad \text{or} \quad \partial_\nu F^{\mu\nu} = \frac{4\pi}{c} j^\mu \qquad \partial^\mu F_{\mu\nu} = -\frac{4\pi}{c} j_\nu \quad \text{or} \quad \partial^\nu F_{\mu\nu} = \frac{4\pi}{c} j_\mu$$

- The conservation of charge easily follows from the above equation and the asymmetric property.

$$\partial_\mu \partial_\nu F^{\mu\nu} = -\partial_\nu \partial_\mu F^{\nu\mu} = -\partial_\mu \partial_\nu F^{\mu\nu} \;\rightarrow\; \partial_\mu \partial_\nu F^{\mu\nu} = 0$$

$$\partial_\nu j^\nu = -\frac{c}{4\pi} \partial_\nu \partial_\mu F^{\mu\nu} = 0$$

- The "internal" Maxwell equations (homogeneous equations):

$$\nabla \cdot \mathbf{B} = 0 \qquad \longrightarrow \qquad \sum_{i=1}^{3} \partial_i B_i = 0 \qquad \longrightarrow \qquad \partial_1 F^{23} + \partial_2 F^{31} + \partial_3 F^{12} = 0$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \longrightarrow \qquad \partial_2 E_3 - \partial_3 E_2 + \partial_0 B_1 = 0 \qquad \longrightarrow \qquad \partial_2 F^{30} + \partial_3 F^{20} + \partial_0 F^{23} = 0$$

$$\boxed{\partial_\mu F^{\nu\sigma} + \partial_\nu F^{\sigma\mu} + \partial_\sigma F^{\mu\nu} = 0} \qquad \text{or} \qquad \partial^\mu F_{\nu\sigma} + \partial^\nu F_{\sigma\mu} + \partial^\sigma F_{\mu\nu} = 0$$

The equation can be written concisely as $F^{[\mu\nu,\sigma]} = 0$ or $F_{[\mu\nu,\sigma]} = 0$, where [ ] around indices denote all permutations of indices, with even permutation contributing with a positive sign and odd permutation with a negative sign, for example,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = A_{[\nu,\mu]}$$

- Transformation of Electromagnetic Fields

- Since $F^{\mu\nu}$ is a second-rank tensor, its components transform in the usual way:

$$F'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} F^{\alpha\beta} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} F^{\alpha\beta}$$

For a pure boost along the x-axis:

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow$$

$$E_x' = F'^{01} = \Lambda^{0}{}_{0}\Lambda^{1}{}_{1}F^{01} + \Lambda^{0}{}_{1}\Lambda^{1}{}_{0}F^{10} = \gamma^2 E_x - \beta^2\gamma^2 E_x = E_x$$

$$E_y' = F'^{02} = \Lambda^{0}{}_{0}\Lambda^{2}{}_{2}F^{02} + \Lambda^{0}{}_{1}\Lambda^{2}{}_{2}F^{12} = \gamma E_y - \beta\gamma B_z$$

$$E_z' = F'^{03} = \Lambda^{0}{}_{0}\Lambda^{3}{}_{3}F^{03} + \Lambda^{0}{}_{1}\Lambda^{3}{}_{3}F^{13} = \gamma E_z + \beta\gamma B_y$$

$$B_x' = F'^{23} = \Lambda^{2}{}_{2}\Lambda^{3}{}_{3}F^{23} = B_x$$

$$B_y' = F'^{31} = \Lambda^{3}{}_{3}\left(\Lambda^{1}{}_{0}F^{30} - \Lambda^{1}{}_{1}F^{31}\right) = \beta\gamma E_z + \gamma B_y$$

$$B_z' = F'^{12} = \Lambda^{1}{}_{0}\Lambda^{2}{}_{2}F^{02} + \Lambda^{1}{}_{1}\Lambda^{2}{}_{2}F^{12} = -\beta\gamma E_y + \gamma B_z$$

- In general,

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \qquad \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}$$

$$\mathbf{E}'_{\perp} = \gamma\left(\mathbf{E}_{\perp} + \boldsymbol{\beta} \times \mathbf{B}\right) \qquad \mathbf{B}'_{\perp} = \gamma\left(\mathbf{B}_{\perp} - \boldsymbol{\beta} \times \mathbf{E}\right)$$

The concept of a pure electric or pure magnetic is not Lorentz invariant.

- Lorenz invariants:

  dot product of F with itself or “square” of F:

$$F^{\mu\nu}F_{\mu\nu} = \sum_{i=1}^{3} F^{0i}F_{0i} + \sum_{i=1}^{3} F^{i0}F_{i0} + \sum_{i\neq j} F^{ij}F_{ij} = 2\left(\mathbf{B}^2 - \mathbf{E}^2\right)$$

  determinant of F:

$$\det F = \left(\mathbf{E}\cdot\mathbf{B}\right)^2$$

[Relativistic Mechanics and the Lorentz Four-Force]

- We can define a **four-acceleration** a^μ in exactly the same way as we obtained the four-velocity.

$$a^\mu \equiv \frac{dU^\mu}{d\tau}$$

Note that the four-acceleration and four-velocity are orthogonal:

$$\vec{a} \cdot \vec{U} \equiv \frac{dU^\mu}{d\tau} U_\mu = \frac{1}{2} \frac{d}{d\tau}\left(U^\mu U_\mu\right) = \frac{1}{2} \frac{d}{d\tau}\left(-c^2\right) = 0$$

- We can also define the four-force F^μ from the Lorentz force, so as to obtain a relativistic form of Newton's equation.

$$\boxed{F^\mu \equiv m_0 a^\mu = \frac{dP^\mu}{d\tau}}$$

$$\vec{F} = \frac{d\vec{P}}{d\tau} = \gamma \frac{d\vec{P}}{dt} = \gamma \left(\frac{1}{c} \frac{dE}{dt}, \frac{d\mathbf{p}}{dt} \right)$$

Since $\mathbf{F}_{\text{Lorentz}} = q\left[\mathbf{E} + \frac{1}{c}(\mathbf{v} \times \mathbf{B})\right]$, the **Lorentz four-force** should involve (1) the electromagnetic field tensor and (2) the four-velocity and should also be (3) a four-vector and (4) proportional to the charge of the particle. Therefore, the simplest possibility is

$$\boxed{F^\mu{}_{\text{Lorentz}} = \frac{q}{c} F^{\mu\nu} U_\nu}$$

- Let's check to see if it is indeed what we want.

$$F^0{}_{\text{Lorentz}} = \frac{q}{c} F^{0\nu} U_\nu = \frac{q}{c} \sum_{i=1}^{3} E_i \gamma v_i = \frac{q}{c} \gamma (\mathbf{E} \cdot \mathbf{v}) \longrightarrow \frac{dE}{dt} = q\mathbf{E} \cdot \mathbf{v} \quad \text{: conservation of energy}$$

The rate of change of particle energy is the mechanical work done on the particle by the field.

$$F^1{}_{\text{Lorentz}} = \frac{q}{c} F^{1\nu} U_\nu = \frac{q}{c} \left(F^{10}(-\gamma c) + F^{12} \gamma v_2 + F^{13} \gamma v_3 \right)$$
$$= \frac{q}{c} \gamma \left(E_1 c + B_3 v_2 - B_2 v_3 \right) \longrightarrow \frac{d\mathbf{p}}{dt} = q \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right)$$

Therefore, we obtained the desired expression for the four-Lorentz force.

- Note that **the four-force is always orthogonal to the four-velocity**:

$$\vec{F} \cdot \vec{U} = m_0 (\vec{a} \cdot \vec{U}) = 0$$

It implies that **every four-force must have some velocity dependence**.

For the Lorentz four-force, in particular, we find

$$\vec{F}_{\text{Lorentz}} \cdot \vec{U} = \frac{q}{c} F^{\mu\nu} U_\mu U_\nu = 0,$$

because $F^{\mu\nu}$ is antisymmetric and $U_\mu U_\nu$ is symmetric.

[Fields of a Uniformly Moving Charge]

- Let's find the fields of a charge moving with constant velocity v along the x axis. In the rest frame of the particle the fields are

$$\mathbf{E}' = \left(E'_x, E'_y, E'_z\right) = \frac{q}{r'^3}\left(x', y', z'\right) \qquad \text{where} \qquad r' = \left(x'^2 + y'^2 + z'^2\right)^{1/2}$$

$$\mathbf{B}' = (0,0,0)$$

inverse transformation of the previous one:

$$\mathbf{E}_\parallel = \mathbf{E}'_\parallel \qquad \mathbf{B}_\parallel = \mathbf{B}'_\parallel$$

$$\mathbf{E}_\perp = \gamma\left(\mathbf{E}'_\perp - \boldsymbol{\beta}\times\mathbf{B}'\right) \qquad \mathbf{B}_\perp = \gamma\left(\mathbf{B}'_\perp + \boldsymbol{\beta}\times\mathbf{E}'\right) \longrightarrow$$

$$E_x = \frac{qx'}{r'^3} \qquad B_x = 0$$

$$E_y = \gamma\frac{qy'}{r'^3} \qquad B_y = -\gamma\beta\frac{qz'}{r'^3}$$

$$E_z = \gamma\frac{qz'}{r'^3} \qquad B_z = \gamma\beta\frac{qy'}{r'^3}$$

Since $x' = \gamma\left(x - vt\right),\ y' = y,\ z' = z$, we obtain

$$E_x = \gamma\frac{q\left(x - vt\right)}{r^3} \qquad B_x = 0$$

$$E_y = \gamma\frac{qy}{r^3} \qquad B_y = -\gamma\beta\frac{qz}{r^3}$$

$$E_z = \gamma\frac{qz}{r^3} \qquad B_z = \gamma\beta\frac{qy}{r^3}$$

where $\quad r = \left[\left(x - vt\right)^2 + y^2 + z^2\right]^{1/2}$

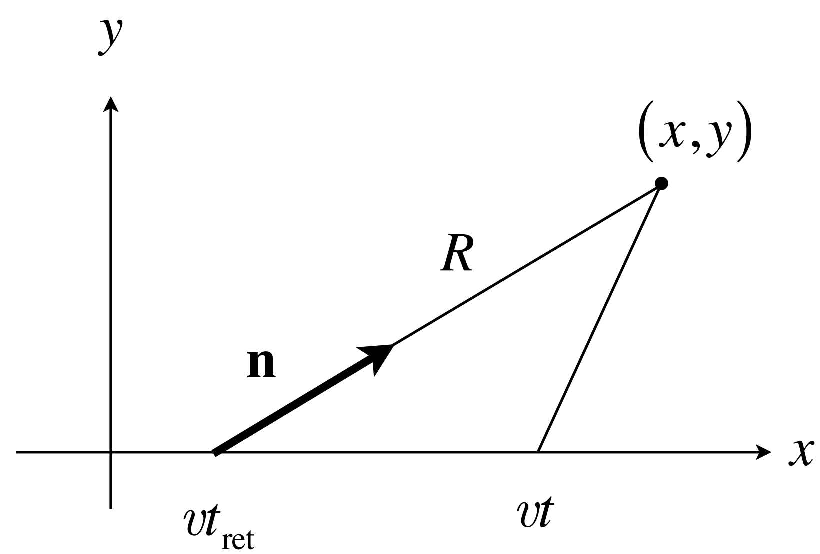
Is this equivalent to the fields given by the Lienard-Wiechert potentials?

- Velocity field from the retarded potential

- For simplicity, assume $z=0$.

$$\mathbf{E}=\left(E_x,E_y,E_z\right)=\gamma\frac{q}{r^3}(x-vt,y,z)$$

$$=\gamma\frac{q}{\left(\gamma^2\bar{x}^2+y^2\right)^{3/2}}(\bar{x},y,0) \quad \text{where} \quad \bar{x}\equiv x-vt$$



Let us first find where the retarded position of the particle is.

$$t_{\text{ret}}\equiv t-R/c$$

$$R^2=\left(x-vt_{\text{ret}}\right)^2+y^2=\left(\bar{x}+\beta R\right)^2+y^2$$

$$\mathbf{n}=\frac{\left(\bar{x}+\beta R\right)}{R}\hat{\mathbf{x}}+\frac{y}{R}\hat{\mathbf{y}}=\left(\frac{\bar{x}}{R}+\beta\right)\hat{\mathbf{x}}+\frac{y}{R}\hat{\mathbf{y}}$$

$$R\longrightarrow \quad (1-\beta^2)^2R^2-2\bar{x}\beta R-\bar{x}^2-y^2=0$$

$$R^2-2\bar{x}\gamma^2\beta R-\gamma^2\left(\bar{x}^2+y^2\right)=0$$

$$R=\gamma^2\beta\bar{x}\pm\left[\gamma^4\beta^2\bar{x}^2+\gamma^2\left(\bar{x}^2+y^2\right)\right]^{1/2}$$

$$=\gamma^2\beta\bar{x}\pm\gamma\left[\gamma^2\beta^2\bar{x}^2+\left(\bar{x}^2+y^2\right)\right]^{1/2}$$

$$=\gamma^2\beta\bar{x}\pm\gamma\left(\gamma^2\bar{x}^2+y^2\right)^{1/2}$$

positive solution $\rightarrow$ $R=\gamma^2\beta\bar{x}+\gamma\left(\gamma^2\bar{x}^2+y^2\right)^{1/2}$

(1) $\mathbf{n}-\boldsymbol{\beta}=\frac{\bar{x}}{R}\hat{\mathbf{x}}+\frac{y}{R}\hat{\mathbf{y}}$

$$\mathbf{E}=\gamma\frac{q}{\left(\gamma^2\bar{x}^2+y^2\right)^{3/2}}\left(\mathbf{n}-\boldsymbol{\beta}\right)$$

(2) $\left(\gamma^2\bar{x}^2+y^2\right)^{1/2}=\frac{R-\gamma^2\beta\bar{x}}{\gamma}=R\gamma\left(\frac{1}{\gamma^2}-\frac{\beta\bar{x}}{R}\right)$

$$=R\gamma\left(1-\beta^2-\frac{\beta\bar{x}}{R}\right)$$

$$=R\gamma\left[1-\beta\left(\frac{\bar{x}}{R}+\beta\right)\right]$$

$$=R\gamma\left(1-\mathbf{n}\cdot\boldsymbol{\beta}\right)=R\gamma\kappa$$

$$\therefore\ \mathbf{E}=q\frac{\left(\mathbf{n}-\boldsymbol{\beta}\right)}{\gamma^2\kappa^3R^2}=q\frac{\left(\mathbf{n}-\boldsymbol{\beta}\right)\left(1-\beta^2\right)}{\kappa^3R^2}$$: velocity field

- Time-dependence of the electric field at a point

- Let us choose the field point to be at $(0,b,0)$.

  This involves no loss in generality. Then,

$$E_x = -\frac{q\gamma vt}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} = -\frac{q}{b^2}\frac{\gamma vt / b}{(\gamma^2 v^2 t^2 / b^2 + 1)^{3/2}}$$

$$E_y = \frac{q\gamma b}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} = \frac{q\gamma}{b^2}\frac{1}{(\gamma^2 v^2 t^2 / b^2 + 1)^{3/2}}$$

$$E_z = 0$$

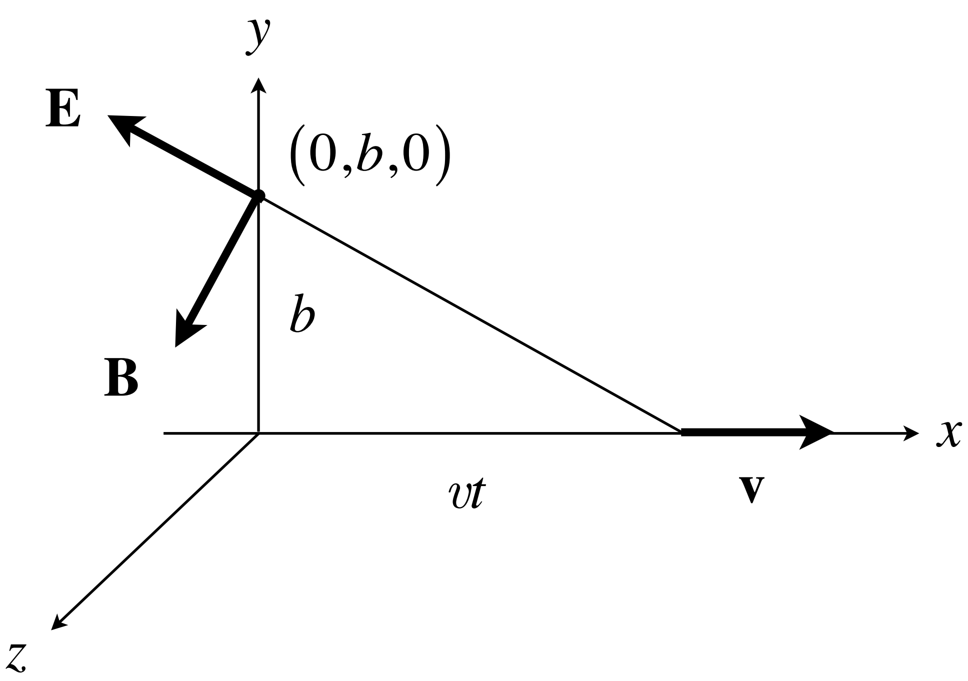
$$B_x = 0$$

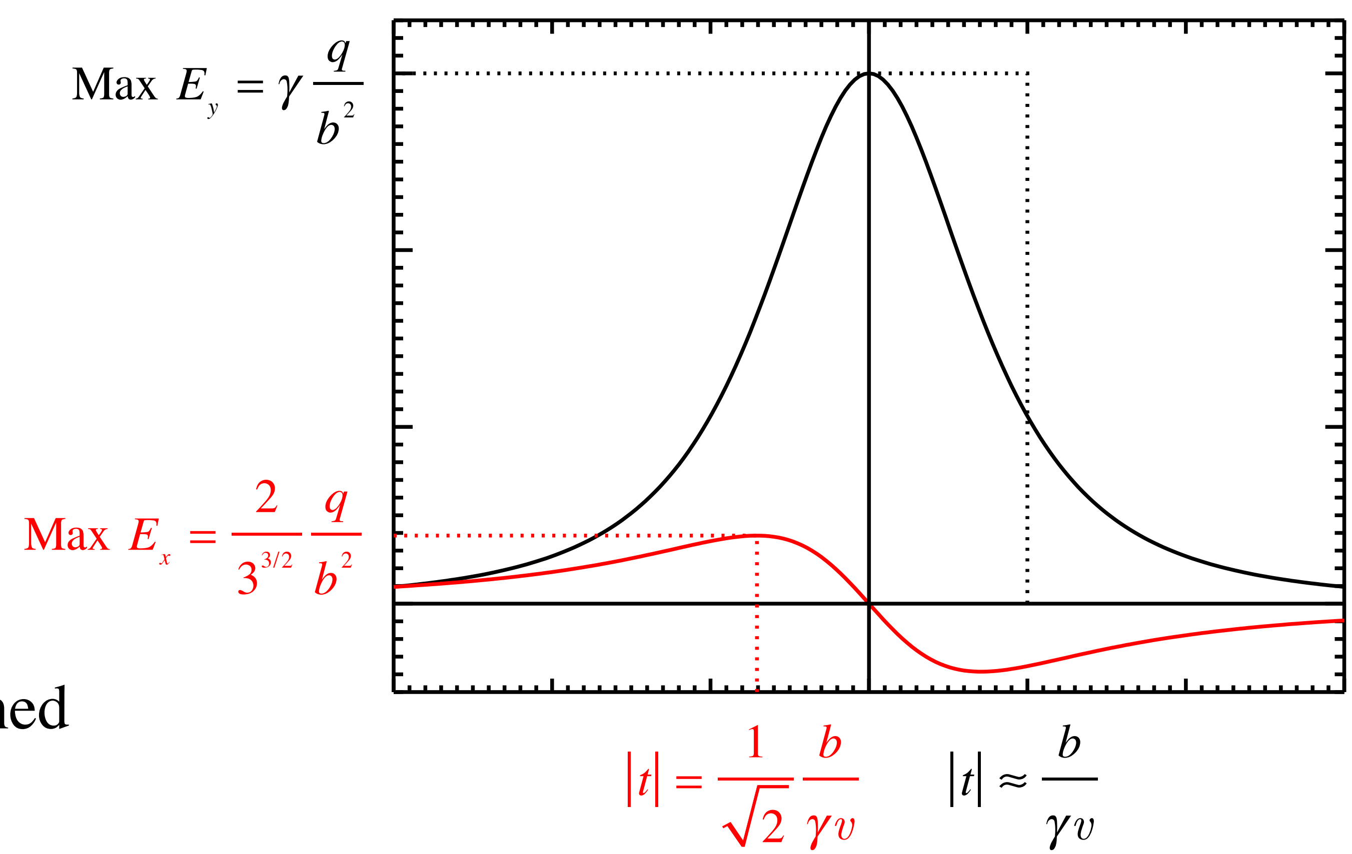
$$B_y = 0$$

$$B_z = \beta E_y$$

As $\gamma \gg 1 \;\rightarrow\; |E_x| \ll E_y$

The field of a highly relativistic charge appears to be a pulse of radiation traveling in the same direction as the charge and confined to the tranverse plane.





- Spectrum of the pulse

- Spectrum of this pulse of virtual radiation.

$$\hat{E}(\omega) = \frac{1}{2\pi}\int E_y(t)e^{i\omega t}\,dt$$

$$= \frac{q\gamma b}{2\pi}\int_{-\infty}^{\infty}\left(\gamma^2 v^2 t^2 + b^2\right)^{-3/2} e^{i\omega t}\,dt = \frac{q\gamma b}{2\pi}\int_{0}^{\infty}\left(\gamma^2 v^2 t^2 + b^2\right)^{-3/2}\left(e^{i\omega t} + e^{-i\omega t}\right)dt$$

$$= \frac{q\gamma b}{\pi}\int_{0}^{\infty}\left(\gamma^2 v^2 t^2 + b^2\right)^{-3/2}\cos\omega t\,dt$$

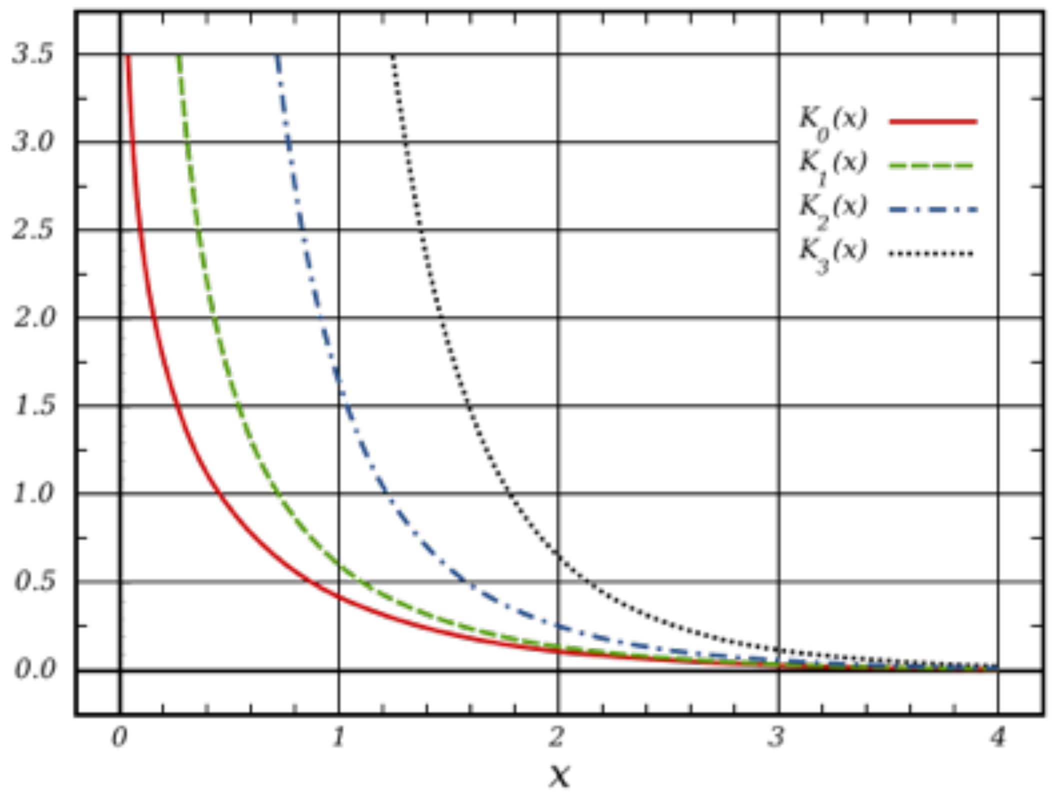
This integral can be done in terms of the modified Bessel function:

$$K_n(x) \equiv \frac{\Gamma(n+1/2)(2x)^n}{\sqrt{\pi}}\int_0^{\infty}\frac{\cos t}{\left(t^2+x^2\right)^{n+1/2}}dt$$

Gamma function: $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$

$$\hat{E}(\omega) = \frac{q\gamma b}{\pi}\left(\frac{\gamma^2 v^2}{\omega^2}\right)^{-3/2}\frac{1}{\omega}\int_0^{\infty}\left(\omega^2 t^2 + \frac{b^2\omega^2}{\gamma^2 v^2}\right)^{-3/2}\cos\omega t\,d\omega t$$

$$= \frac{q}{\pi b v}\frac{b\omega}{\gamma v}K_1\left(\frac{b\omega}{\gamma v}\right)$$

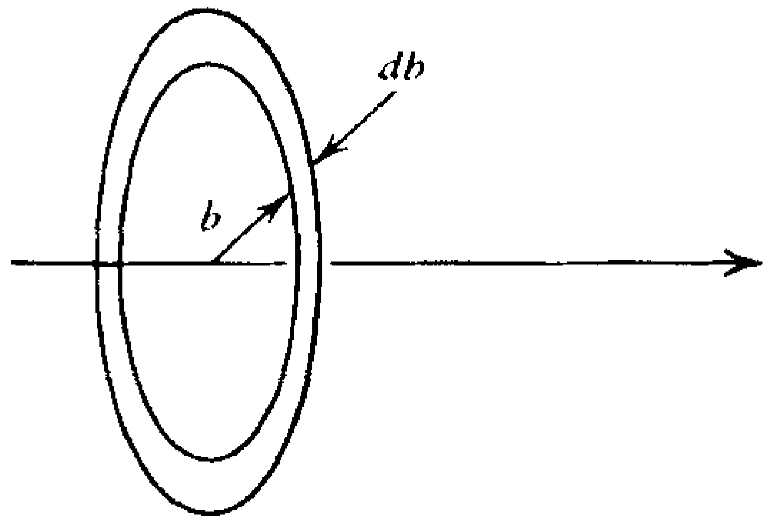


Thus the spectrum is

$$\frac{dW}{dA\,d\omega} = c\left|\hat{E}(\omega)\right|^2 = \frac{q^2}{\pi^2 b^2 v^2}\left(\frac{b\omega}{\gamma v}\right)^2 K_1^{\,2}\left(\frac{b\omega}{\gamma v}\right)$$

The spectrum starts cut off for $\omega > \gamma v / b$.

$$\Delta\omega \sim \frac{1}{\Delta t} \sim \gamma v / b$$



- Total energy per unit frequency range is obtained by

$$\frac{dW}{d\omega} = 2\pi \int_{b_{\min}}^{b_{\max}} \frac{dW}{dA\,d\omega} b\,db$$

The lower limit has been chosen as some minimum distance, such that the approximation of the field by means of classical electrodynamics and a point charge is valid.

$$\frac{dW}{d\omega} = \frac{2q^2c}{\pi v^2} \int_x^{\infty} y K_1^{\,2}(y)\,dy \qquad \text{where} \quad y \equiv \frac{\omega b}{\gamma v}, \ \text{and} \quad x \equiv \frac{\omega b_{\min}}{\gamma v}$$

$$= \frac{2q^2c}{\pi v^2}\left[x K_0(x) K_1(x) - \frac{1}{2} x^2 \left(K_1^{\,2}(x) - K_0^{\,2}(x) \right) \right]$$

- Two limiting cases:

(1) $\omega \ll \dfrac{\gamma v}{b_{\min}} \quad (x \ll 1)$,

$$xK_0(x)K_1(x) - \frac{1}{2}x^2\left(K_1{}^2(x) - K_0{}^2(x)\right)$$
$$\approx x\left(-\ln(x/2) - \gamma\right)\frac{1}{x} - \frac{x^2}{2}\left[\frac{1}{x^2} - \left(\ln(x/2) + \gamma\right)^2\right] \longrightarrow \frac{dW}{d\omega} = \frac{2q^2 c}{\pi v^2}\ln\left(0.68\,\frac{\gamma v}{\omega b_{\min}}\right)$$
$$\approx \ln\left[\frac{2}{x}e^{-(\gamma+1/2)}\right]$$
$$= \ln\left(\frac{0.68}{x}\right)$$

(2) $\omega \gg \dfrac{\gamma v}{b_{\min}} \quad (x \gg 1)$,

$$xK_0(x)K_1(x) - \frac{1}{2}x^2\left(K_1{}^2(x) - K_0{}^2(x)\right) \longrightarrow \frac{dW}{d\omega} = \frac{q^2 c}{2v^2}\exp\left(-\frac{2\omega b_{\min}}{\gamma v}\right)$$
$$\approx x\frac{\pi}{2x}e^{-2x} - \frac{1}{2}x^2\frac{\pi}{2x}e^{-2x}\left[\left(\frac{3}{8x}\right)^2 - \left(\frac{1}{8x}\right)^2\right]$$
$$= \frac{\pi}{4}e^{-2x}$$

[Emission from Relativistic Particles]

- Total emitted power:

Imagine **an instantaneous rest frame** *K'*, such that the particle has zero velocity at a certain time. We can then calculate the radiation emitted by use of the dipole (Larmor) formula.

Suppose that the particle emits a total amount of energy dW' in this frame in time dt'. The momentum of this radiation is zero, $d\mathbf{p}' = 0$, because the emission is symmetrical in the frame.

The energy in a frame *K* moving with velocity $-\mathbf{v}$ w.r.t. the particle is:

$$dW = \gamma dW' \quad \longleftarrow \quad dE = cdP^0 = c\tilde{\Lambda}^0_\mu dP'^\mu = c\tilde{\Lambda}^0_0 dP'^0 = \gamma dE'$$

The time interval dt is simply

$$dt = \gamma dt'$$

The total power emitted in frames *K* and *K'* are given by

$$P = \frac{dW}{dt}, \qquad P' = \frac{dW'}{dt'}$$

Thus **the total emitted power is a Lorentz invariant** for any emitter that emits with front-back symmetry in its instantaneous rest frame.

$$P = P'$$

- **the Larmor formula in covariant form:**

  Recall that $\vec{a}\cdot\vec{U}=0$, and because $\vec{U}=(c,\mathbf{0})$ in the instantaneous rest frame of the particle, we have

$$a_0'=0 \quad \rightarrow \quad |\mathbf{a}'|^2 = a_k' a'^k = a_\mu' a'^\mu = \vec{a}\cdot\vec{a}$$

  Therefore,

$$P'=\frac{2q^2}{3c^3}|\mathbf{a}'|^2 \quad \longrightarrow \quad P=\frac{2q^2}{3c^3}\vec{a}\cdot\vec{a}$$

- Expression of P in terms of the three-vector acceleration

  Recall

$$dt=\gamma\left(dt'+\frac{v}{c^2}dx'_{\parallel}\right)$$

$$u_{\parallel}=\frac{u_{\parallel}'+v}{1+vu_{\parallel}'/c^2}$$

$$u_{\perp}=\frac{u_{\perp}'}{\gamma(1+vu_{\parallel}'/c^2)}$$

$\longrightarrow$

$$\sigma\equiv(1+vu_{\parallel}'/c^2)$$

$$dt=\gamma dt'\sigma$$

$$u_{\parallel}=\frac{u_{\parallel}'+v}{\sigma}$$

$$u_{\perp}=\frac{u_{\perp}'}{\gamma\sigma}$$

$\longrightarrow$

$$dt=\gamma dt'\sigma$$

$$du_{\parallel}=\frac{du_{\parallel}'}{\sigma}-\frac{u_{\parallel}'+v}{\sigma^2}\frac{v}{c^2}du_{\parallel}'$$

$$=\frac{du_{\parallel}'}{\sigma^2}\left(1-\frac{v^2}{c^2}\right)=\frac{du_{\parallel}'}{\gamma^2\sigma^2}$$

$$du_{\perp}=\frac{du_{\perp}'}{\gamma\sigma}-\frac{u_{\perp}'}{\gamma\sigma^2}\frac{v}{c^2}du_{\parallel}'$$

$$=\frac{1}{\gamma\sigma^2}\left(\sigma du_{\perp}'-\frac{vu_{\perp}'}{c^2}du_{\parallel}'\right)$$

Hence,

$$a_\parallel = \frac{du_\parallel}{dt} = \frac{1}{\gamma^3\sigma^3}\frac{du'_\parallel}{dt'}$$

$$a_\perp = \frac{du_\perp}{dt} = \frac{1}{\gamma^2\sigma^3}\left(\sigma\frac{du'_\perp}{dt'} - \frac{vu'_\perp}{c^2}\frac{du'_\parallel}{dt'}\right)$$

$\longrightarrow$

Transformation of three-vector acceleration:

$$a_\parallel = \frac{1}{\gamma^3\sigma^3}a'_\parallel$$

$$a_\perp = \frac{1}{\gamma^2\sigma^3}\left(\sigma a'_\perp - \frac{vu'_\perp}{c^2}a'_\parallel\right)$$

where $\sigma \equiv \left(1 + \frac{vu'_\parallel}{c^2}\right)$

In an instantaneous rest frame of a particle,

$$u'_\parallel = u'_\perp = 0,\ \ \sigma = 1$$

$$a'_\parallel = \gamma^3 a_\parallel$$
$$a'_\perp = \gamma^2 a_\perp$$

Note $\tan\theta'_a \equiv \frac{a'_\perp}{a'_\parallel} = \frac{1}{\gamma}\frac{a_\perp}{a_\parallel} = \frac{1}{\gamma}\tan\theta_a$

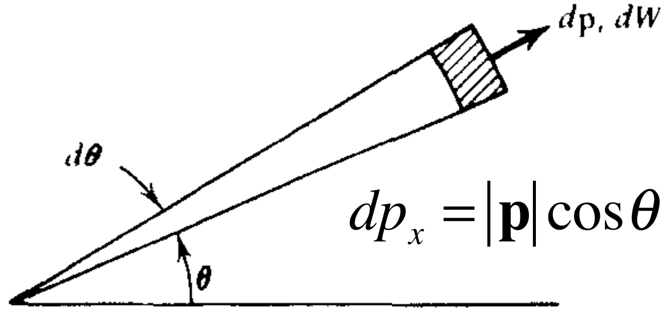
Thus we can write

$$P = \frac{2q^2}{3c^3}|\mathbf{a}'|^2 = \frac{2q^2}{3c^3}\left(a'^2_\parallel + a'^{\,2}_\perp\right)$$

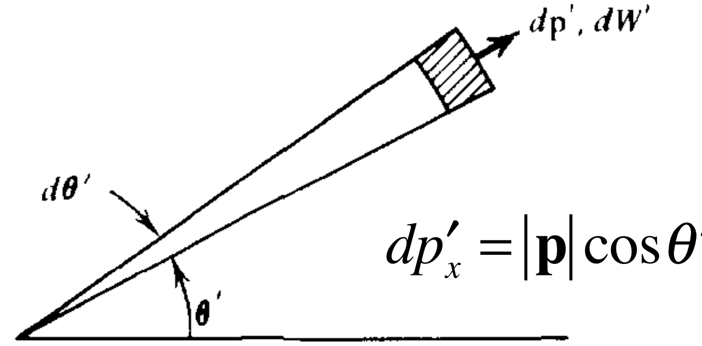
$\longrightarrow$

$$P = \frac{2q^2}{3c^3}\gamma^4\left(\gamma^2 a_\parallel^{\,2} + a_\perp^{\,2}\right)$$

- <u>Angular Distribution of Emitted and Received Power</u>



$dp_x = |\mathbf{p}|\cos\theta$



$dp_x' = |\mathbf{p}|\cos\theta'$

Note:

$d\phi' = d\phi$

In the instantaneous rest frame of the particle, let us consider an amount of energy *dW'* that is emitted into the solid angle $d\Omega' = \sin\theta' d\theta' d\phi'$ (see the above figure).

$$\mu \equiv \cos\theta \rightarrow d\Omega = d\mu d\phi \qquad \mu' \equiv \cos\theta' \rightarrow d\Omega' = d\mu' d\phi'$$

Recall $\cos\theta = \dfrac{\cos\theta' + \beta}{1+\beta\cos\theta'} \rightarrow \mu = \dfrac{\mu' + \beta}{1+\beta\mu'}$, or inverse $\mu' = \dfrac{\mu - \beta}{1-\beta\mu}$

$$d\mu = \frac{d\mu'}{1+\beta\mu'} - \frac{\mu'+\beta}{(1+\beta\mu')^2}\beta d\mu' \longrightarrow$$

$$d\mu = \frac{d\mu'}{\gamma^2(1+\beta\mu')^2},\quad d\mu = \gamma^2(1-\beta\mu)^2 d\mu'$$

$$d\Omega = \frac{d\Omega'}{\gamma^2(1+\beta\mu')^2},\quad d\Omega = \gamma^2(1-\beta\mu)^2 d\Omega'$$

- Power

Recall that energy and momentum form a four-vector

$$P^{\mu} = \left(\frac{E}{c}, \mathbf{p}\right), \text{ and } |\mathbf{p}| = \frac{E}{c} \quad \longrightarrow \quad dW = \gamma\left(dW' + v dp'_x\right) = \gamma\left(1 + \beta\mu'\right) dW'$$

$$\therefore \; dW = \gamma\left(1 + \beta\mu'\right) dW', \; dW = \gamma^{-1}\left(1 - \beta\mu\right)^{-1} dW'$$

$$\frac{dW}{d\Omega} = \gamma^{3}\left(1 + \beta\mu'\right)^{3} \frac{dW'}{d\Omega'}, \; \frac{dW}{d\Omega} = \gamma^{-3}\left(1 - \beta\mu\right)^{-3} \frac{dW'}{d\Omega'}$$

In the rest frame, the power emitted in a unit time interval is $\frac{dP'}{d\Omega'} \equiv \frac{dW'}{dt' d\Omega'}$

However, in the observer's frame, there are two possible choices for the time interval to calculate the power.

(1) $dt = \gamma dt'$:

This is the time interval during which the emission occurs. With this choice we obtain **the emitted power**.

(2) $dt_A = \gamma(1 - \beta\mu) dt'$, or $dt_A = \gamma^{-1}(1 + \beta\mu')^{-1} dt'$:

This is the time interval of the radiation as received by a stationary observer in K. With this choice we obtain **the received power**.

- Thus we obtain the two results:

$$\frac{dP_e}{d\Omega} = \gamma^2 (1+\beta\mu')^3 \frac{dP'}{d\Omega'} = \gamma^{-4} (1-\beta\mu)^{-3} \frac{dP'}{d\Omega'}$$
$$\frac{dP_r}{d\Omega} = \gamma^4 (1+\beta\mu')^4 \frac{dP'}{d\Omega'} = \gamma^{-4} (1-\beta\mu)^{-4} \frac{dP'}{d\Omega'}$$

P_r is the power actually measured by an observer. It has the expected symmetry property of yielding the inver transformation by interchanging primed and unprimed variables, along with a change of sign of β.

P_e is used in the discussion of emission coefficient.

In practice, the distinction between emitted and received power is often not important, since they are equal in an average sense for stationary distributions of particles.

- Beaming effect:

If the radiation if isotropic in the particle's frame, then the angular distribution in the observer's frame will be highly peaked in the forward direction for highly relativistic velocities.

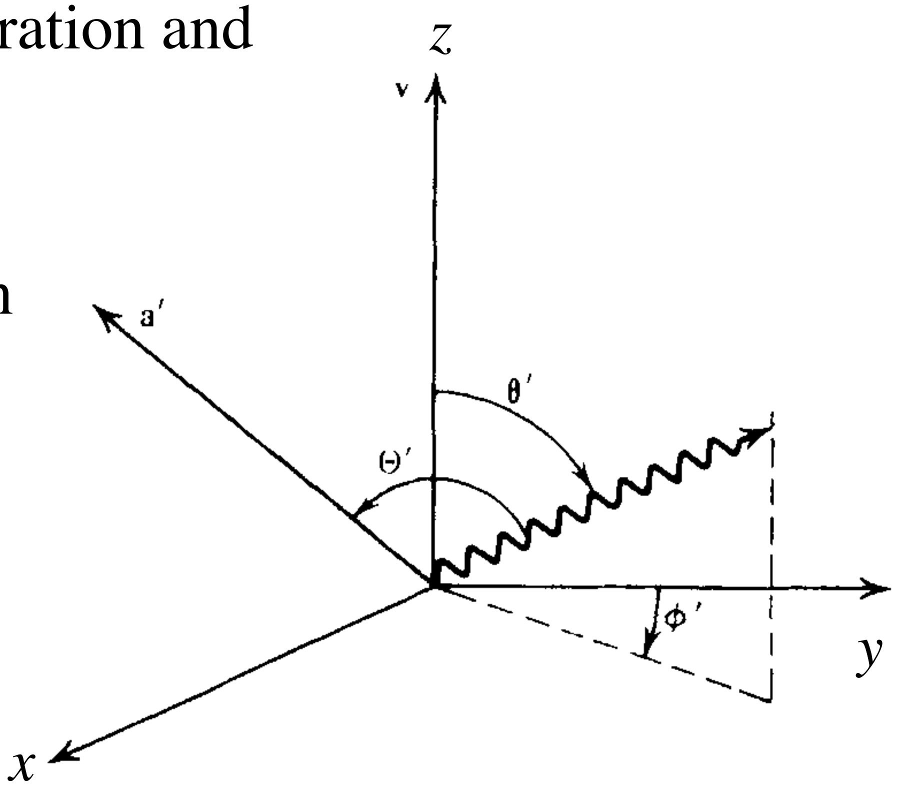
The factor $\gamma^{-4}(1-\beta\mu)^{-4}$ is sharply peaked near $\theta \approx 0$ with an angular scale of order $1/\gamma$.

$$\gamma^{-4}(1-\beta\mu)^{-4} \approx \gamma^{-4}\left[1-\left(1-\frac{1}{2\gamma^2}\right)\left(1-\frac{\theta^2}{2}\right)\right]^{-4} = \gamma^{-4}\left(\frac{1}{2\gamma^2}+\frac{\theta^2}{2}\right)^{-4} = \left(\frac{2\gamma}{1+\gamma^2\theta^2}\right)^4$$

- Dipole emission from a slowly moving particle

$$\frac{dP'}{d\Omega'} = \frac{q^2 a'^2}{4\pi c^3}\sin^2\Theta'$$

$\Theta' =$ the angle between the acceleration and the direction of emission.



Using $a'_\parallel = \gamma^3 a_\parallel,\ a'_\perp = \gamma^2 a_\perp$ and $\frac{dP_r}{d\Omega} = \gamma^{-4}(1-\beta\mu)^{-4}\frac{dP'}{d\Omega'}$, we obtain

$$\boxed{\frac{dP_r}{d\Omega} = \frac{q^2}{4\pi c^3}\frac{\left(\gamma^2 a_\parallel^2 + a_\perp^2\right)}{(1-\beta\mu)^4}\sin^2\Theta'}$$

To use this formula, we must relate Θ' to the angles in K.

(1) Acceleration parallel to velocity: $\Theta' = \theta',\ a_\perp = 0$

$$\sin^2\Theta' = 1-\mu'^2 = 1-\left(\frac{\mu-\beta}{1-\beta\mu}\right)^2 = \frac{1-\mu^2}{\gamma^2(1-\beta\mu)^2} \longrightarrow \frac{dP_\parallel}{d\Omega} = \frac{q^2 a_\parallel^2}{4\pi c^3}\frac{1-\mu^2}{(1-\beta\mu)^6}$$

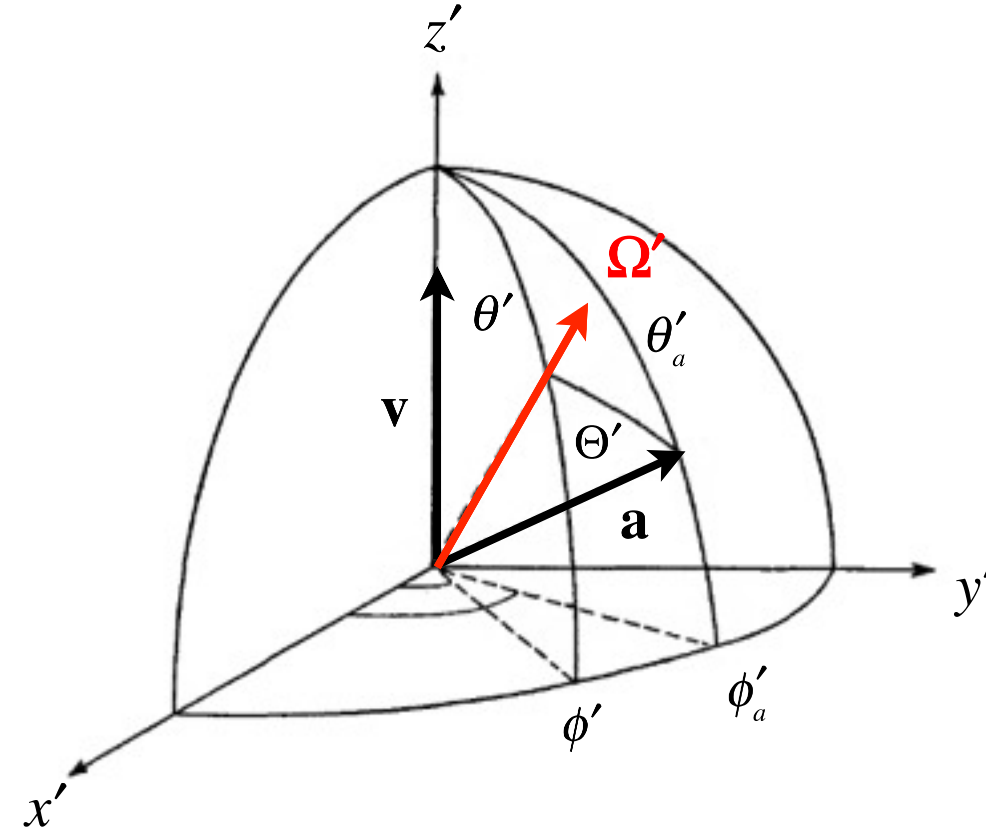
(2) Acceleration perpendicular to velocity: $\cos\Theta' = \sin\theta'\cos\phi',\ a_\parallel = 0$ (when a is in y-direction in the above figure)

$$\sin^2\Theta' = 1-\frac{\left(1-\mu^2\right)\cos^2\phi}{\gamma^2(1-\beta\mu)^2} \longrightarrow \frac{dP_\perp}{d\Omega} = \frac{q^2 a_\perp^2}{4\pi c^3}\frac{1}{(1-\beta\mu)^4}\left[1-\frac{\left(1-\mu^2\right)\cos^2\phi}{\gamma^2(1-\beta\mu)^2}\right]$$

(3) In general

$$\cos\Theta' = \mu'\mu_a' + \left(1-\mu'^2\right)^{1/2}\left(1-\mu_a'^2\right)^{1/2}\cos\left(\phi'-\phi_a'\right)$$

See Eq. (219) in Chadrasekhar (1960)

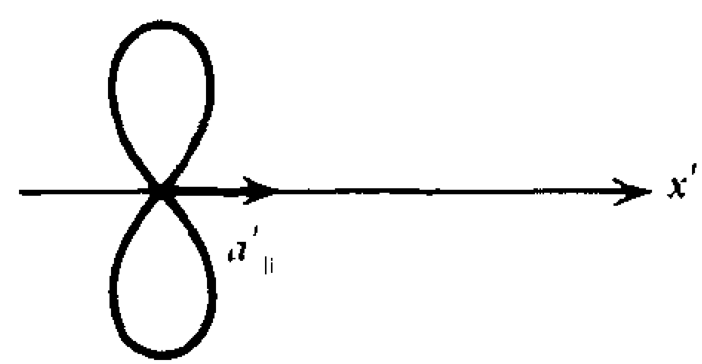


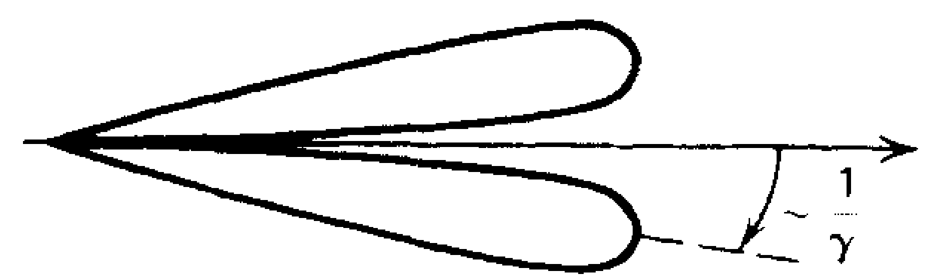
- In the extreme relativistic limit, the radiation becomes strongly peaked in the forward direction.

particle's rest frame:

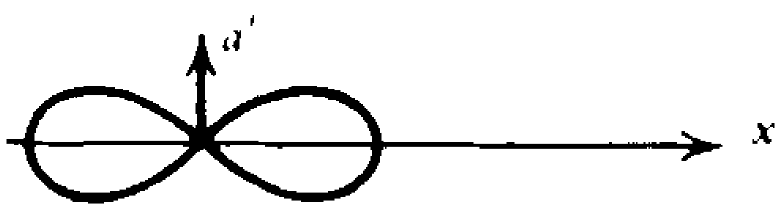
observer's frame:

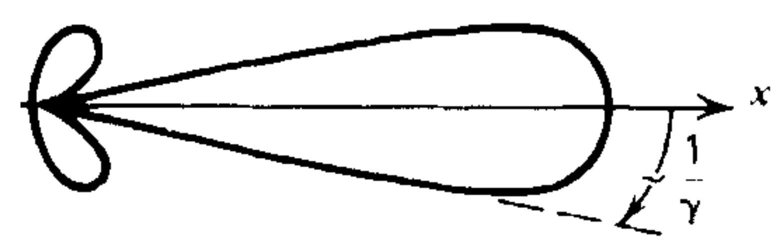
parallel acceleration:





perpendicular acceleration:





[Invariant Phase Volumes and Specific Intensity]

- **Phase volume**

Consider a group of particles that occupy a slight spread in position and in momentum at a particular time. In a rest frame comoving with the particles, they occupy a spatial volume element and a momentum volume element.

$$d^3\mathbf{x}' = dx'dy'dz'$$
$$d^3\mathbf{p}' = dp'_x dp'_y dp'_z$$

$\longrightarrow$ phase volume in the comoving frame:

$$d\mathcal{V}' \equiv d^3\mathbf{x}' d^3\mathbf{p}' = dx'dy'dz'dp'_x dp'_y dp'_z$$

In the observer's frame, $dx = \gamma^{-1}dx', \;\; dy = dy', \;\; dz = dz'$

$$dp_x = \gamma\left(dp'_x + \beta dP'_0\right), \;\; dp_y = dp'_y, \;\; dp_z = dp'_z$$

We note that $dP'_0 = 0 + \mathcal{O}(dp'^2_x)$ because the velocities are near zero in the comoving frame and the energy is quadratic in velocity. Therefore, we have

$dp_x = \gamma dp'_x$ and $\boxed{d\mathcal{V}' \equiv d^3\mathbf{x}' d^3\mathbf{p}' = d^3\mathbf{x}\, d^3\mathbf{p} \equiv d\mathcal{V}}$: Lorentz invariant

This contains no reference to particle mass, and therefore it has applicability to photons.

The phase space density

$$f \equiv \frac{dN}{d\mathcal{V}}$$

is an invariant, since the number of particles within the phase volume element is a countable quantity and itself invariant.

- **Specific Intensity and Source Function**

  Definition of the energy density per unit solid angle per frequency range.

$$h\nu f p^2 dp d\Omega = U_\nu(\Omega) d\Omega d\nu$$

  Since $p = h\nu / c$ and $U_\nu(\Omega) = I_\nu / c$ we find that

$$\boxed{\frac{I_\nu}{\nu^3} = \text{Lorentz invariant}}$$

  Because the source function occurs in the transfer equation as the difference $I_\nu - S_\mu$, the source function must have the same transformation properties as the intensity.

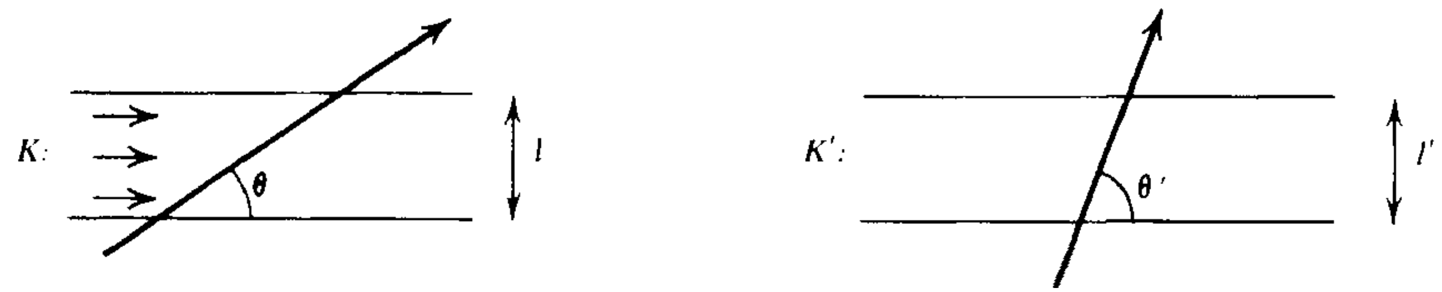
- **Optical Depth, Absorption Coefficient and Emission Coefficient**

  The optical depth must be an invariant, since $e^{-\tau}$ gives the fraction of photons passing through the material, and this involves simple counting.

$$\boxed{\tau = \text{Lorentz invariant}}$$

- **Absorption Coefficient and Emission Coefficient**

Consider the optical depth in two frames:



Then, the optical depth is

$$\tau = \frac{l\alpha_\nu}{\sin\theta} = \frac{l}{\nu\sin\theta}\nu\alpha_\nu = \text{Lorentz invariant}$$

Note that $\nu\sin\theta$ is proportional to the y component of the photon four-momentum $\vec{k} = \left(\frac{\omega}{c}, \mathbf{k}\right)$.

Both k_y and l are the same in both frames, being perpendicular to the motion. Therefore, we have

$$\boxed{\nu\alpha_\nu = \text{Lorentz invariant}}$$

Finally, we obtain the transformation of the emission coefficient from the definition of the source function: $S_\nu \equiv j_\nu / \alpha_\nu$

$$\boxed{\frac{j_\nu}{\nu^2} = \text{Lorentz invariant}}$$