Radiative Processes in Astrophysics

Lecture 6
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Raman Scattering

- If the energy of the internal state (E_0) is less than that of the incoming photon $(h\nu)$, then a scattered photon of energy $h\nu - E_0$ can be produced.

  Raman scattering or the Raman effect is the inelastic scattering of a photon.

  When photons are scattered from an atom or molecule, most photons are elastically scattered (i.e., Rayleigh scattering), such that the scattered photons have the same energy (frequency and wavelength) as the incident photons. However, a small fraction of the scattered photons (approximately 1 in 10 million) are scattered by an excitation, with the scattered photons having a frequency different from, and usually lower than, that of the incident photons.

- Scattering of the O VI doublet (λλ1038, 1032) by neutral hydrogen.

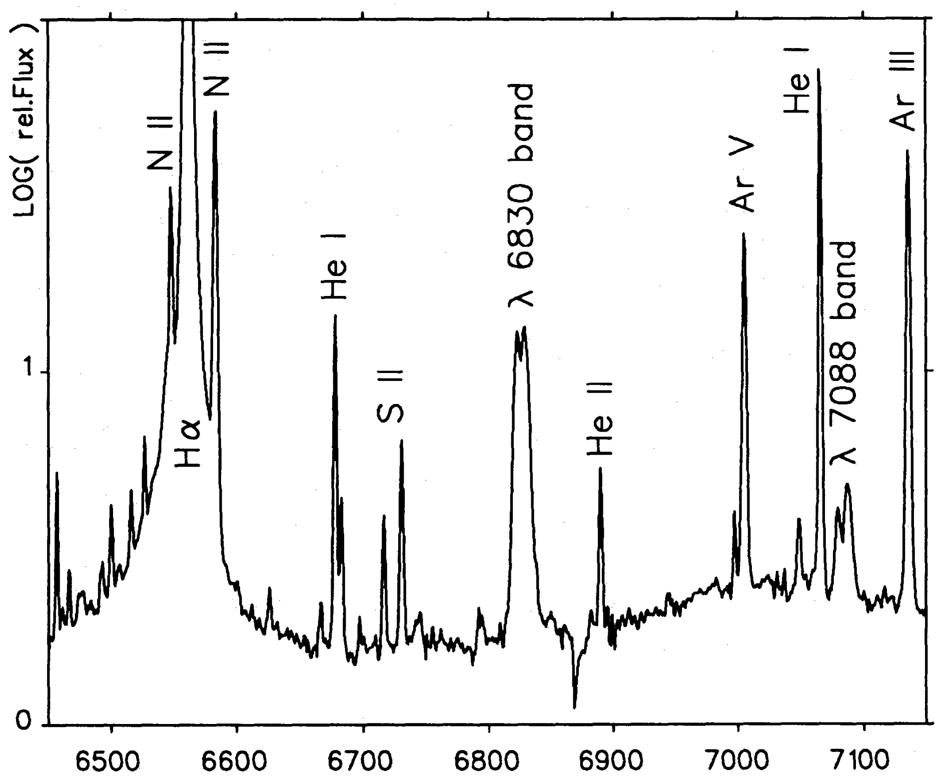
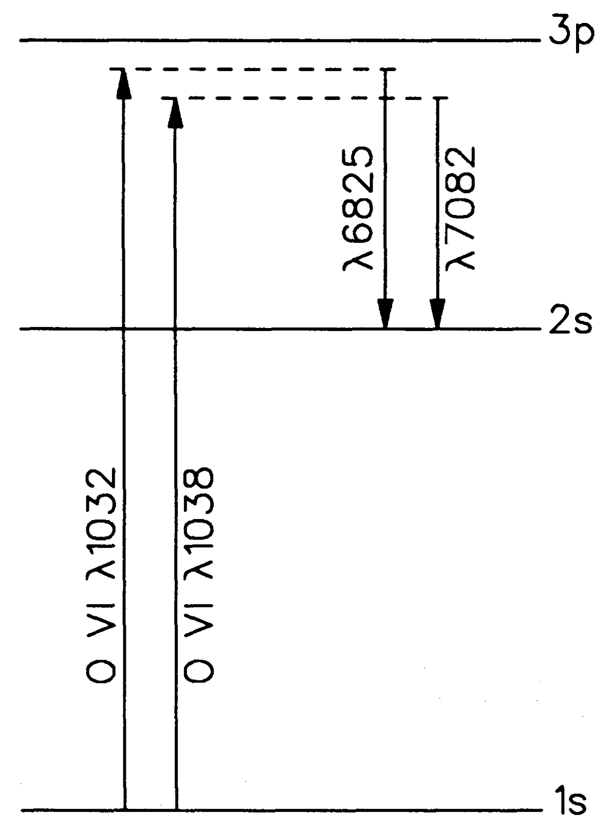


Fig.1. Raman scattered emission bands in the symbiotic star V1016 Cyg. The spectrum was obtained on the 1.93m telescope at the Observatoire de Haute Provence.

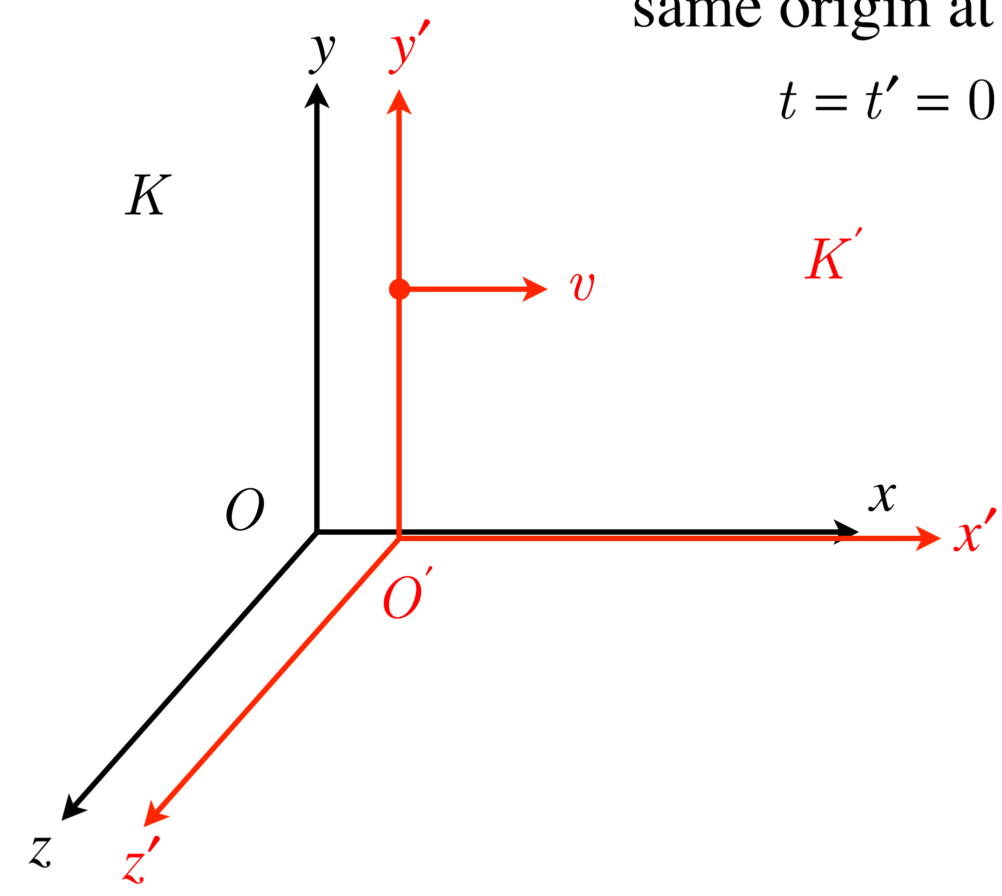


Schmid (1989, A&A, 211, L31)

Relativistic Covariance and Kinematics

Galilean Transformation/Relativity

- Galilean transformation is used to transform between the coordinates of two **inertial frames of reference** which differ only by constant relative motion within the constructs of Newtonian physics.



$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

Newton's law is invariant under the Galilean transformation.

However, Maxwell'e equations are not invariant under the Galilean transformation.

- Lorentz transformation is the result of attempts by Lorentz and others to explain how the speed of light was observed to be independent of the reference frame, and to understand the symmetries of the Maxwell's equations.

* Review of Lorentz Transformations *

- **Postulates in the special theory of relativity**

  (1) The laws of nature are the same in two frames of reference in uniform relative motion with no rotation.

  (2) The speed of light is c in all such frames.

- **space-time event**: an event that takes place at a location in space and time.

- **Derivation of Lorentz transforms**:

  If a pulse of light is emitted at the origin at $t = 0$, each observer will see an expanding sphere centered on his own origin. Therefore, we have the equations of the expanding sphere in each frame.

$$x^2 + y^2 + z^2 - c^2t^2 = 0, \quad x'^2 + y'^2 + z'^2 - c^2t'^2 = 0 \qquad (1)$$

  Since space is assumed to be homogeneous, the transformation must be linear.

$$x' = a_1 x + a_2 t, \quad y' = y, \quad z' = z, \quad t' = b_1 x + b_2 t$$

  We note that the origin of K' $(x' = 0)$ is a point that moves with speed v as seen in K. Its location in K is given by $x = vt$. Therefore, we have

$$\frac{a_2}{a_1} = -v \qquad \begin{aligned} x' &= a_1(x - vt) \\ y' &= y \\ z' &= z \\ t' &= b_1 x + b_2 t \end{aligned} \qquad (2)$$

Substitute Eqs. (2) into Eq. (1): $x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2$

$$a_1{}^2(x - vt)^2 + y^2 + z^2 - c^2(b_1x + b_2t)^2 = x^2 + y^2 + z^2 - c^2t^2$$

$$(a_1{}^2 - c^2b_1{}^2)x^2 - 2(a_1{}^2v + c^2b_1b_2)xt + (a_1{}^2v^2 - c^2b_2{}^2)t^2 + y^2 + z^2 = x^2 + y^2 + z^2 - c^2t^2$$

(Note: we didn't assume that $x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$)

Therefore, the following equations should be satisfied.

$$a_1{}^2 - c^2b_1{}^2 = 1 \quad (a)$$

$$(a_1{}^2v + c^2b_1b_2) = 0 \quad (b)$$

$$a_1{}^2v^2 - c^2b_2{}^2 = -c^2 \quad (c)$$

$$\longrightarrow \quad (a)\quad b_1{}^2 = \frac{a_1{}^2 - 1}{c^2} \qquad (c)\quad b_2{}^2 = 1 + \frac{v^2}{c^2}a_1{}^2$$

$$(b)\quad a_1{}^4v^2 = -c^4b_1{}^2b_2^2 = c^2a_1{}^2 + v^2a_1{}^4 - c^2 - v^2a_1{}^2$$

$$\rightarrow \quad a_1 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \equiv \gamma$$

$$\longrightarrow \quad (a)\quad b_1 = \gamma, \quad (c)\quad b_2 = -\frac{v}{c^2}\gamma$$

Finally, we obtain the Lorentz transformation (and its inverse):

$$x' = \gamma(x - vt) \qquad x = \gamma(x' + vt')$$

$$y' = y \qquad y = y'$$

$$z' = z \qquad z = z'$$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \qquad t = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

where $\gamma \equiv \dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \left(1 - \beta^2\right)^{-1/2}$; $\beta \equiv \dfrac{v}{c}$

Lorentz factor $1 \le \gamma < \infty$; $0 \le \beta < 1$

Length Contraction / Time Dilation

- **Length contraction** (Lorentz-Fitzgerald contraction): Suppose a rigid rod of length $L_0 = x_2' - x_1'$ is carried at rest in K'. What is the length as measured in K? The positions of the ends of the rod are marked <u>at the same time in K</u>.

$$L_0 = x_2' - x_1' = \gamma(x_2 - x_1) = \gamma L$$
$$L = L_0 / \gamma$$

Therefore, the rod appears shorter by a factor $1/\gamma$ in K.

If both carry rods (of the same length when compared at rest) each thinks the other's rod has shrunk!

It would appear to K' that the two ends of the moving stick were not marked at the same time by the other observer (in K).

- **Time dilation**: Suppose <u>a clock at rest at the origin of K'</u> measures off a time interval $T_0 = t_2' - t_1'$. What is the time interval measured in K? Note that the clock is at rest at the origin of K' so that $x_2' = x_1' = 0$.

$$T = t_2 - t_1 = \gamma(t_2' - t_1') = \gamma T_0$$
$$T = \gamma T_0$$

The time interval has increased by a factor γ, so that the moving clock appears to have slowed down.

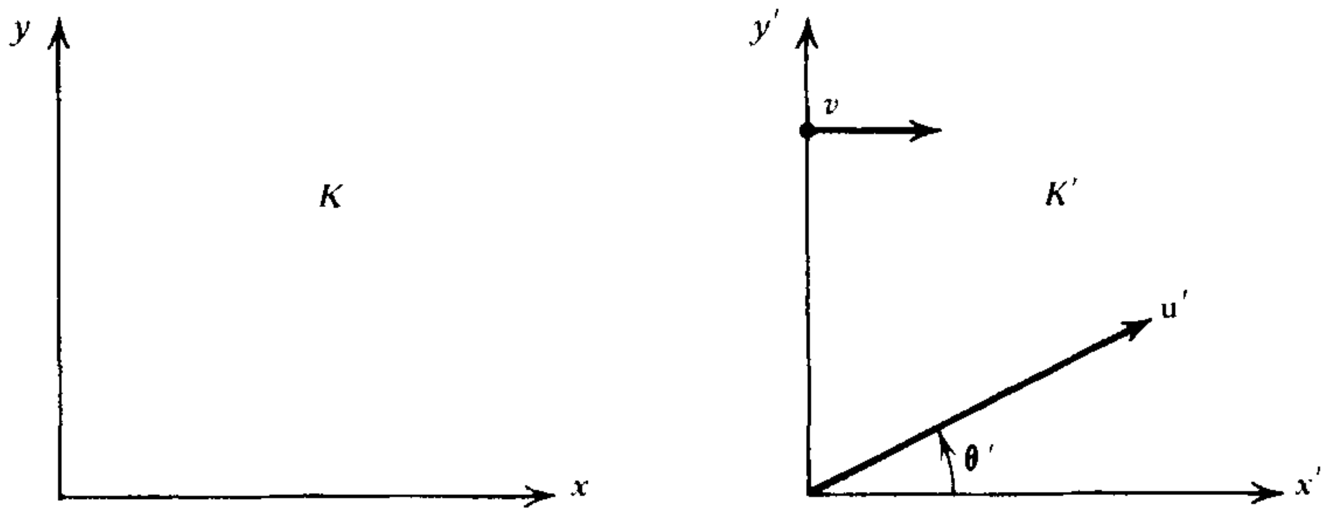
Time dilation is detected in the increased half-lives of unstable particles moving rapidly in an accelerator or in the cosmic-ray flux.

Transformation of Velocities

- Simultaneity is relative: Simultaneous events at two different spatial points in the primed frame is not simultaneous in the unprimed frame.
- If a point has a velocity $\mathbf{u}'$ in frame K', what is its velocity $\mathbf{u}$ in frame K. Writing Lorentz transformations for differentials

$$dx = \gamma(dx' + v dt'), \quad dy = dy', \quad dz = dz'$$

$$dt = \gamma\left(dt' + \frac{v}{c^2}dx'\right)$$



$$u_x = \frac{dx}{dt} = \frac{\gamma(dx' + v dt')}{\gamma(dt' + v dx'/c^2)} = \frac{u_x' + v}{1 + v u_x'/c^2}$$

$$u_y = \frac{dy}{dt} = \frac{dy'}{\gamma(dt' + v dx'/c^2)} = \frac{u_y'}{\gamma(1 + v u_x'/c^2)}$$

$$u_z = \frac{dz}{dt} = \frac{u_z'}{\gamma(1 + v u_x'/c^2)}$$

or

$$u_{\parallel} = \frac{u_{\parallel}' + v}{1 + v u_{\parallel}'/c^2}$$

$$u_{\perp} = \frac{u_{\perp}'}{\gamma(1 + v u_{\parallel}'/c^2)}$$

- Aberration formula: the directions of the velocities in the two frames are related by

$$\tan\theta = \frac{u_\perp}{u_\parallel} = \frac{u_\perp{}'}{\gamma(u_\parallel{}' + v)} = \frac{u'\sin\theta'}{\gamma(u'\cos\theta' + v)} \qquad \text{where } u' \equiv |\mathbf{u}'|.$$

- **Aberration of light**

For the case of light: $u' = c$

$$\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' + v/c)} = \frac{\sin\theta'}{\gamma(\cos\theta' + \beta)}$$

$$\cos\theta = \frac{\gamma(\cos\theta' + v/c)}{\sqrt{\gamma^2(\cos\theta' + v/c)^2 + \sin^2\theta'}} = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}$$

$$\sin\theta = \frac{\sin\theta'}{\sqrt{\gamma^2(\cos\theta' + v/c)^2 + \sin^2\theta'}} = \frac{\sin\theta'}{\gamma\left(1 + \beta\cos\theta'\right)}$$

Using the identity, $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$

The aberration formula can be written as: $\tan\left(\frac{\theta}{2}\right) = \frac{(1/\gamma)\sin\theta'}{1 + \beta\cos\theta' + \cos\theta' + \beta} = \frac{(1/\gamma)\sin\theta'}{(1+\beta)(1+\cos\theta')}$

$$\tan\left(\frac{\theta}{2}\right) = \left(\frac{1-\beta}{1+\beta}\right)^{1/2} \tan\left(\frac{\theta'}{2}\right) \quad \rightarrow \quad \theta < \theta'$$

- **Beaming ("headlight") effect**:

If photons are emitted isotropically in K', then half will have $\theta' < \pi/2$ and half $\theta' > \pi/2$.

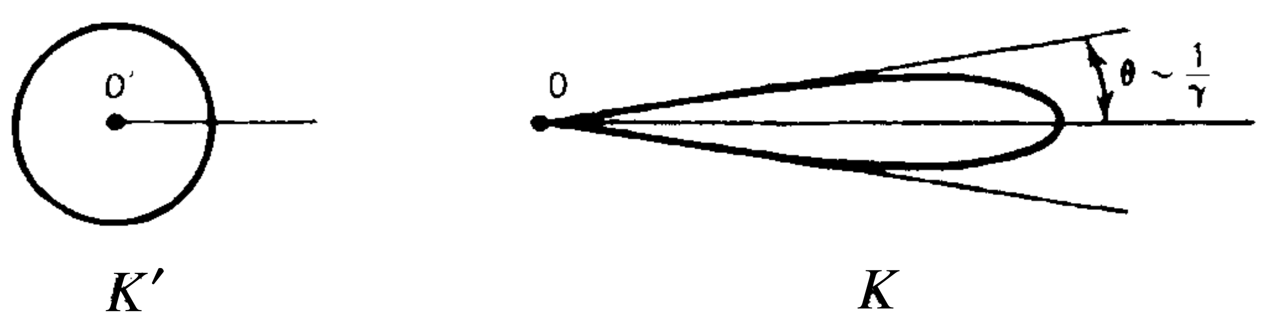
Consider a photon emitted at right angles to v in K'. Then we have

$$\text{beam half-angle:} \quad \sin\theta_b = \frac{1}{\gamma}, \quad \cos\theta_b = \beta, \quad \text{or} \quad \tan\left(\frac{\theta_b}{2}\right) = \left(\frac{1-\beta}{1+\beta}\right)^{1/2}$$

For highly relativistic speeds, $\gamma \gg 1$, θ_b becomes small:

$$\theta_b \sim \frac{1}{\gamma}$$

Therefore, in frame K, photons are concentrated in the forward direction, with half of them lying within a cone of half-angle $1/\gamma$. Very few photons will be emitted $\theta \gg 1/\gamma$.

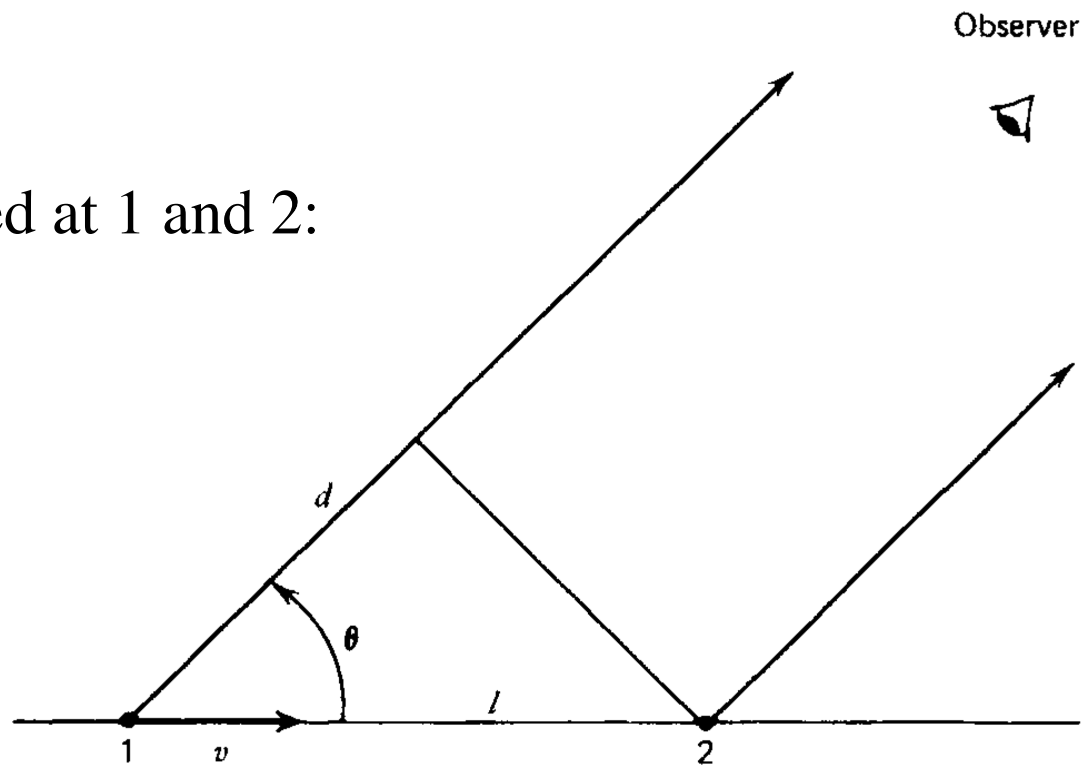


Doppler Effect

- In the rest frame of the observer K, imagine that the moving source emits one period of radiation as it moves from point 1 to point 2 at velocity v.

Let frequency of the radiation in the rest frame (K') of the source = ω'. Then the time taken to move from point 1 to point 2 in the observer's frame is given by the time-dilation effect:

$$\Delta t = \Delta t' \gamma = \frac{2\pi}{\omega'} \gamma$$



Difference in arrival times Δt_A of the radiation emitted at 1 and 2:

$$\Delta t_A = \Delta t - \frac{d}{c} = \Delta t \left(1 - \beta \cos\theta\right)$$

Therefore, the observed frequency ω will be

$$\omega = \frac{2\pi}{\Delta t_A} = \frac{\omega'}{\gamma(1-\beta\cos\theta)}, \quad \text{or} \quad \boxed{\frac{\omega}{\omega'} = \frac{1}{\gamma(1-\beta\cos\theta)}}$$

Note $1-\beta\cos\theta$ appears even classically. The factor γ^{-1} is purely a relativistic effect.

Transverse (or second-order) Doppler effect :

$$\frac{\omega}{\omega'} = \frac{1}{\gamma} \le 1 \quad \text{at} \quad \theta = \pi/2$$

- Beam half-angle: $\sin\theta_b = \gamma^{-1}$
- Angle for null Doppler shift:

$$\frac{\omega}{\omega'} = \frac{1}{\gamma(1-\beta\cos\theta_n)} = 1$$

$$\rightarrow \quad \cos\theta_n = \frac{1-\gamma^{-1}}{\beta} = \left(\frac{1-\gamma^{-1}}{1+\gamma^{-1}}\right)^{1/2}$$

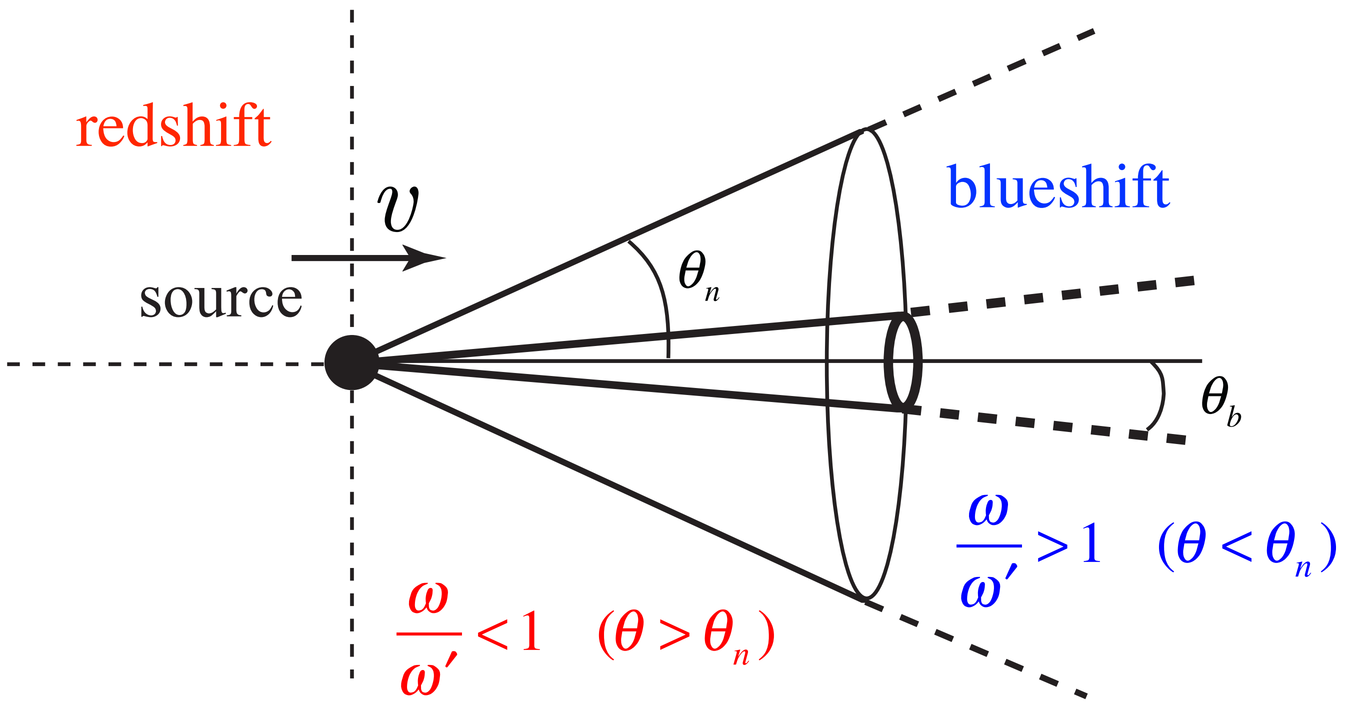
Relativistic Doppler effect can yield redshift even as a source approaches.

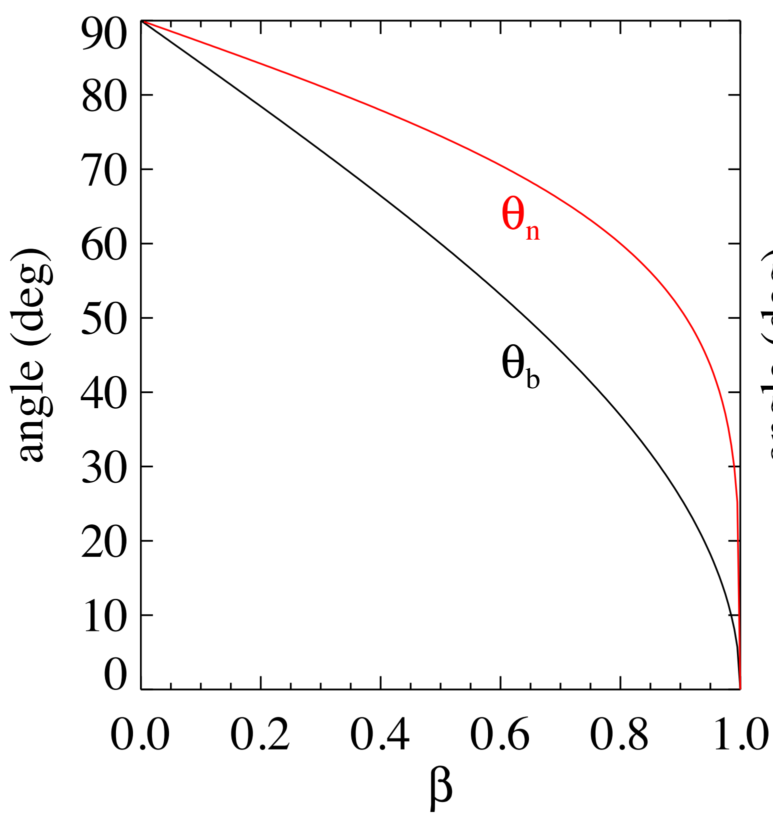
$$\cos\theta_n = \left(\frac{1-\gamma^{-1}}{1+\gamma^{-1}}\right)^{1/2} \approx 1-\frac{1}{\gamma} \quad \text{for } \gamma \gg 1$$

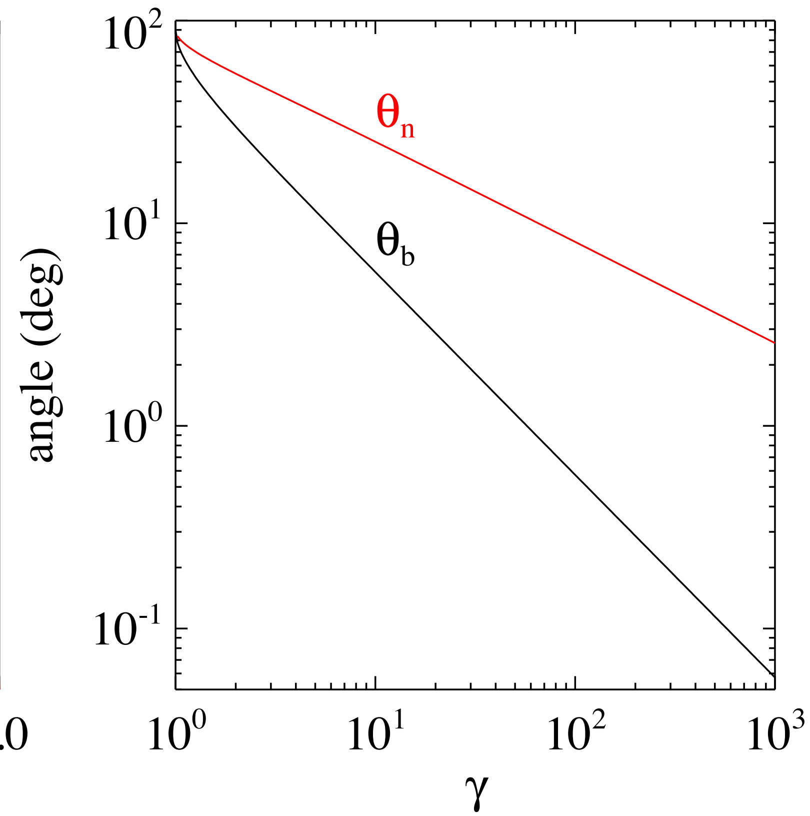
$$1-\frac{\theta_n^{\ 2}}{2} \approx 1-\frac{1}{\gamma}$$

$$\therefore \quad \theta_n \approx \sqrt{\frac{2}{\gamma}} \approx \sqrt{2\theta_b}$$

- Note $\theta_b \le \theta_n$







Lorentz Invariant

- **Lorentz invariant**: A quantity (scalar) that remains unchanged by a Lorentz transform is said to be a "Lorentz invariant."

$$
\begin{aligned}
x'^2 + y'^2 + z'^2 - c^2 t'^2 &= \gamma^2 (x - \beta ct)^2 + y^2 + z^2 - \gamma^2 (ct - \beta x)^2 \\
&= \gamma^2 (1 - \beta^2) x^2 + y^2 + z^2 + \gamma^2 (\beta^2 c^2 - c^2) t^2 \\
&= x^2 + y^2 + z^2 - c^2 t^2
\end{aligned}
$$

- Proper distance: Since all events are subject to the same transformation, the space-time "interval" between two event is also invariant.

$$ds^2 \equiv dx^2 + dy^2 + dz^2 - c^2 dt^2$$

  This is the spatial distance between two events occurring at the same time. This is called the proper distance between the two points.

- Proper time (interval):

$$c^2 d\tau^2 \equiv -ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$$

  This measures time intervals between events occurring at the same spatial location $(dx = dy = dz = 0)$

  If the coordinate differentials refer to the position of the origin of another reference frame traveling with velocity v, then $d\tau = dt\left(1 - \beta^2\right)^{1/2} = dt / \gamma$

  This is the time dilation formula in which $d\tau$ is the time interval measured by the frame in motion.

* Four-Vectors *

- **Four-vector**: Invariant in 3D rotations: $dx^2 + dy^2 + dz^2$

  By analogy, the invariance of the space-time interval suggests to define a vector in 4D space (4 dimensional space-time vector or four-vector). The quantities x^μ $(\mu = 0,1,2,3)$ define coordinates of an event in space-time.

$$\vec{x} \equiv x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix}$$

  Contravariant components

- Minkowski space: Space-time is not a Euclidean space; it is called Minkowski space.

  Minkowski metric:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  Note that this metric is symmetric: $\eta_{\mu\nu} = \eta_{\nu\mu}$

- **Summation convention**:

  The invariant can now be written in terms of the Minkowski metric: $s^2 = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \eta_{\mu\nu} x^\mu x^\nu$

  In any single term containing **a Greek index repeated twice** (between contravariant and covariant indices), a summation is implied over that index (originated by Einstein). This index is often called a dummy index.

$$s^2 = \eta_{\mu\nu} x^\mu x^\nu$$

  Note that $\eta_{\mu\mu} x^\mu$ is regarded as meaningless.

- **Contravariant/Covariant components**

contravariant components: (superscripted) $x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$

covariant components: (subscripted) $x_\mu = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -ct \\ x \\ y \\ z \end{pmatrix}$

They are related by

$$x_\mu = \eta_{\mu\nu} x^\nu, \quad x^\mu = \eta^{\mu\nu} x_\nu$$
$$s^2 = x^\mu x_\mu$$

The metric can be used to raise or lower indices.

The components of a position (velocity etc.) vector *contra-vary* with a change of basis vectors to compensate. Transformation rules between the following two vector components are inverse. This is the basic idea of "contravariant" and "covariant."

$$x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} x^\nu, \quad \frac{\partial A}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial A}{\partial x^\nu}$$

- **Lorentz transform** (corresponding to a boot along the *x* axis)

transformation matrix:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Lorentz transformation:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$
$$\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}$$

Any arbitrary Lorentz transformation can be written in the above form, since the spatial 3D rotation necessary to align the *x* axes before and after the boost are also of linear form.

- **Conditions for the Lorentz transformation**:

From the invariance of s^2, we must have

$$\eta_{\mu\nu}x^{\mu}x^{\nu} = \eta_{\sigma\tau}x'^{\sigma}x'^{\tau} = \eta_{\sigma\tau}\Lambda^{\sigma}{}_{\mu}\Lambda^{\tau}{}_{\nu}x^{\mu}x^{\nu}$$

This can be true for arbitrary x^{μ} only if

$$\eta_{\mu\nu} = \Lambda^{\sigma}{}_{\mu}\Lambda^{\tau}{}_{\nu}\eta_{\sigma\tau} \quad \text{or equivalently} \quad \eta = \Lambda^{T}\eta\Lambda \quad \text{in matrix form}$$

Taking determinants yields

$$\det \Lambda = \pm 1$$

Proper Lorentz transformations (to keep the right-handness), which rules out reflections.

$$\det \Lambda = 1$$

Isochronous Lorentz transformations (to ensure that the sense of flow of time is the same in frames)

$$\Lambda^{0}{}_{0} \geq 1$$

- **Lorentz transformation** of the covariant component

$$x'_{\mu} = \eta_{\mu\tau}x'^{\tau} = \eta_{\mu\tau}\Lambda^{\tau}{}_{\sigma}x^{\sigma} = \eta_{\mu\tau}\Lambda^{\tau}{}_{\sigma}\eta^{\sigma\nu}x_{\nu}$$

$$\therefore x'_{\mu} = \widetilde{\Lambda}_{\mu}{}^{\nu}x_{\nu} \quad \text{where} \quad \widetilde{\Lambda}_{\mu}{}^{\nu} \equiv \eta_{\mu\tau}\Lambda^{\tau}{}_{\sigma}\eta^{\sigma\nu}$$

$$\widetilde{\Lambda}_{\mu}{}^{\nu} = \frac{\partial x'_{\mu}}{\partial x_{\nu}}$$

- From the invariance of $s^2 = x^\mu x_\mu$:

$$x'^\sigma x'_\sigma = \Lambda^\sigma{}_\nu x^\nu \widetilde{\Lambda}_\sigma{}^\mu x_\mu = \Lambda^\sigma{}_\nu \widetilde{\Lambda}_\sigma{}^\mu x^\nu x_\mu$$

$$\therefore \Lambda^\sigma{}_\nu \widetilde{\Lambda}_\sigma{}^\mu = \delta^\mu{}_\nu$$

$$\therefore \widetilde{\Lambda}_\sigma{}^\mu = \left(\Lambda^{-1}\right)^\mu{}_\sigma$$

where we have introduced the Kronecker delta

$$\delta^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ Identity matrix

- For any arbitrary contravariant components,

$$Q^\mu = \delta^\mu{}_\nu Q^\nu$$

- Note that

$$\eta^{\mu\sigma}\eta_{\sigma\nu} = \delta^\mu{}_\nu$$

- Inverse transform

$$\widetilde{\Lambda}_\sigma{}^\mu \times \left(x'^\sigma = \Lambda^\sigma{}_\nu x^\nu\right) \quad \rightarrow \quad x^\mu = \widetilde{\Lambda}_\sigma{}^\mu x'^\sigma$$

note: $\widetilde{\Lambda}_\sigma{}^\mu = \left(\Lambda^{-1}\right)^\mu{}_\sigma$

Other Four-vectors

- Four-vector:

$$\vec{A} \rightarrow \quad \text{contravariant} \quad A^\mu = \eta^{\mu\nu} A_\nu \qquad \text{covariant} \quad A_\mu = \eta_{\mu\nu} A^\nu$$

$$A'^\mu = \Lambda^\mu{}_\nu A^\nu \qquad A'_\mu = \widetilde{\Lambda}_\mu{}^\nu A_\nu$$

- Consider two four-vectors $\vec{A}$ and $\vec{B}$

$$A'^\mu B'_\mu = \Lambda^\mu{}_\nu \widetilde{\Lambda}_\mu{}^\sigma A^\nu B_\sigma = \delta^\sigma{}_\nu A^\nu B_\sigma = A^\nu B_\nu \quad \rightarrow \quad \boxed{\vec{A}\cdot\vec{B} = A^\mu B_\mu = A'^\mu B'_\mu}$$

Therefore, the scalar product of any two four-vectors is a Lorentz invariant or scalar. In particular, the "square" of a four vector is an invariant. Thus, our starting point, the invariance of $s^2 = x^\mu x_\mu$, is seen to be a general property of four-vectors.

- Note

$$\vec{A}\cdot\vec{A} > 0 \quad \rightarrow \quad \text{spacelike four-vector}$$
$$= 0 \quad \rightarrow \quad \text{light-like (or null) four-vector}$$
$$< 0 \quad \rightarrow \quad \text{time-like four-vector}$$

$A^0 \rightarrow$ time component

$A^i \rightarrow$ space-components (ordinary three-vector)

$$\vec{A}\cdot\vec{B} = -A^0 B^0 + \mathbf{A}\cdot\mathbf{B} = -A^0 B^0 + A^i B_i \quad (i = 1,2,3)$$

Four-velocity

The (infinitesimally small) difference between the coordinates of two events is also a four-vector. Dividing by the proper time yields a four-vector, the four-velocity:

$$U^\mu \equiv \frac{dx^\mu}{d\tau} \longrightarrow \begin{aligned} U^0 &= \frac{cdt}{d\tau} = c\gamma_u \\ U^i &= \frac{dx^i}{d\tau} = \gamma_u u^i \end{aligned}$$

or $\boxed{\vec{U} = \gamma_u \begin{pmatrix} c \\ \mathbf{u} \end{pmatrix}}$ where $\gamma_u \equiv \left(1 - u^2/c^2\right)^{-1/2}, \quad u \equiv \left|\frac{d\mathbf{x}}{dt}\right|$

length of the four-velocity : $\boxed{\vec{U} \cdot \vec{U} = U^\mu U_\mu = -\left(\gamma_u c\right)^2 + \left(\gamma_u \mathbf{u}\right)^2 = -c^2}$

Transformation of the four-velocity:

$$\begin{aligned} U'^0 &= \gamma(U^0 - \beta U^1) \\ U'^1 &= \gamma(-\beta U^0 + U^1) \\ U'^2 &= U^2 \\ U'^3 &= U^3 \end{aligned} \longrightarrow \begin{aligned} \gamma_{u'} c &= \gamma(c\gamma_u - \beta\gamma_u u^1) \\ \gamma_{u'} u'^1 &= \gamma(-\beta c\gamma_u + \gamma_u u^1) \\ \gamma_{u'} u'^2 &= \gamma_u u^2 \\ \gamma_{u'} u'^3 &= \gamma_u u^3 \end{aligned} \longrightarrow \begin{aligned} \gamma_{u'} &= \gamma\gamma_u (1 - vu^1/c^2) \\ \gamma_{u'} u'^1 &= \gamma\gamma_u (u^1 - v) \end{aligned}$$

$\longrightarrow$

velocity component: $u'^1 = \dfrac{u^1 - v}{1 - vu^1/c^2}$ This is the previously derived formula.

speed: $\gamma_{u'} = \gamma\gamma_u \left(1 - \dfrac{vu^1}{c^2}\right)$

Momentum and Energy

- Four-momentum of a particle with a mass m_0 is defined by

$$P^\mu \equiv m_0 U^\mu \qquad P^0 = m_0 c \gamma_v$$
$$P^i = \gamma_v m_0 \mathbf{v}$$

- In the nonrelativistic limit,

$$P^0 c = m_0 c^2 \gamma = m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = m_0 c^2 + \frac{1}{2} m_0 v^2 + \cdots$$

Therefore, we interpret $E \equiv P^0 c = \gamma_v m_0 c^2$ as the total energy of the particle.

The quantity $m_0 c^2$ is interpreted as the rest energy of the particle.

Then,

$$\mathbf{p} \equiv \gamma_v m_0 \mathbf{v}, \quad P^\mu = (E/c, \mathbf{p})$$

Since $\vec{U}^2 = -c^2$, we obtain $\vec{P}^2 = -m_0^2 c^2 = -\dfrac{E^2}{c^2} + |\mathbf{p}|^2$

$$E^2 = m_0^2 c^4 + c^2 |\mathbf{p}|^2$$

- Photons are massless, but we can still define

$$P^\mu = (E/c, \mathbf{p}), \quad E = |\mathbf{p}| c \quad \to \quad \vec{P}^2 = 0$$

Wavenumber vector and frequency

- Quantum relations:

$$E = h\nu = \hbar\omega \qquad \left(\begin{array}{c} \omega = 2\pi\nu \\ k = 2\pi/\lambda \end{array}\right)$$
$$p = E/c = \hbar k$$

We can define four wavenumber vector:

$$\vec{k} \equiv \frac{1}{\hbar}\vec{P} = \left(\frac{\omega}{c}, \mathbf{k}\right)$$

Note that it's a null vector:

$$\vec{k}\cdot\vec{k} = |\mathbf{k}|^2 - \omega^2/c^2 = 0$$

Then, we obtain an invariant:

$$\vec{k}\cdot\vec{x} = k_\mu x^\mu = \mathbf{k}\cdot\mathbf{x} - \omega t$$

Therefore, the phase of the plane wave is an invariant.

- Transform for $\vec{k}$ (Doppler formula):

$$k'^0 = \gamma(k^0 - \beta k^1) \longrightarrow \omega' = \gamma(\omega - \beta c k^1) = \omega\gamma\left(1 - \frac{v}{c}\cos\theta\right)$$
$$k'^1 = \gamma(-\beta k^0 + k^1)$$
$$k'^2 = k^2$$
$$k'^3 = k^3$$

$$\uparrow \; k^1 = (\omega/c)\cos\theta$$

* Tensor Analysis *

- Definition:

  zeroth-rank tensor : Lorentz invariant (scalar) $s' = s$

  first-rank tensor : four-vector $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$

  second-rank tensor: $T'^{\mu\nu} = \Lambda^{\mu}{}_{\sigma}\Lambda^{\nu}{}_{\tau}T^{\sigma\tau}$

- Covariant components and mixed components:

$$T_{\mu\nu} = \eta_{\mu\sigma}\eta_{\nu\tau}T^{\sigma\tau} \qquad T^{\mu}{}_{\nu} = \eta_{\nu\tau}T^{\mu\tau} \qquad T_{\mu}{}^{\nu} = \eta_{\mu\sigma}T^{\sigma\nu}$$

- Transformation rules:

$$\begin{aligned} T'_{\mu\nu} &= \eta_{\mu\alpha}\eta_{\nu\beta}T'^{\alpha\beta} \\ &= \eta_{\mu\alpha}\eta_{\nu\beta}\Lambda^{\alpha}{}_{\gamma}\Lambda^{\beta}{}_{\delta}T^{\gamma\delta} \\ &= \eta_{\mu\alpha}\eta_{\nu\beta}\Lambda^{\alpha}{}_{\gamma}\Lambda^{\beta}{}_{\delta}\eta^{\gamma\sigma}\eta^{\delta\tau}T_{\sigma\tau} \\ &= \widetilde{\Lambda}_{\mu}{}^{\sigma}\widetilde{\Lambda}_{\nu}{}^{\tau}T_{\sigma\tau} \end{aligned}$$

$$\begin{aligned} T'^{\mu}{}_{\nu} &= \eta_{\nu\alpha}T'^{\mu\alpha} \\ &= \eta_{\nu\alpha}\Lambda^{\mu}{}_{\sigma}\Lambda^{\alpha}{}_{\delta}T^{\sigma\delta} \\ &= \eta_{\nu\alpha}\Lambda^{\mu}{}_{\sigma}\Lambda^{\alpha}{}_{\delta}\eta^{\delta\tau}T^{\sigma}{}_{\tau} \\ &= \Lambda^{\mu}{}_{\sigma}\widetilde{\Lambda}_{\nu}{}^{\tau}T^{\sigma}{}_{\tau} \end{aligned}$$

$$\begin{aligned} T'^{\nu}_{\mu} &= \eta_{\mu\alpha}T'^{\alpha\nu} \\ &= \eta_{\mu\alpha}\Lambda^{\alpha}{}_{\beta}\Lambda^{\nu}{}_{\tau}T^{\beta\tau} \\ &= \eta_{\mu\alpha}\Lambda^{\alpha}{}_{\beta}\Lambda^{\nu}{}_{\tau}\eta^{\beta\sigma}T_{\sigma}{}^{\tau} \\ &= \widetilde{\Lambda}_{\mu}{}^{\beta}\Lambda^{\nu}{}_{\tau}T_{\sigma}{}^{\tau} \end{aligned}$$

- symmetric tensor = a tensor that is invariant under a permutation of its indices.

$$T^{\mu\nu} = T^{\nu\mu}$$

- antisymmetric tensor : if it alternates sign when any two indices of the subset are interchanged.

$$T^{\mu\nu} = -T^{\nu\mu}$$

- Examples of the second-rank tensors

A product of two vectors: $A^{\mu} B^{\nu}$

$$A'^{\mu} B'^{\nu} = \Lambda^{\mu}{}_{\sigma} \Lambda^{\nu}{}_{\tau} A^{\sigma} B^{\tau}$$

The Minkowski metric: $\eta^{\mu\nu}$

The Kronecker-delta: $\delta^{\mu}{}_{\nu}$

- Higher-rank tensors
  - Addition: $A^\mu + B^\mu, \quad F^{\mu\nu} + G^{\mu\nu}$
  - Multiplication: $A^\mu B^\nu, \quad F^{\mu\nu} G_{\sigma\tau}$
  - Raising and Lowering Indices: The metric can be used to change contravariant indices into covariant ones, and vice versa, by the processes of raising and lowering.
  - Contraction: $A^\mu B_\nu \quad \rightarrow \quad A^\mu B_\mu$ scalar

    $T^{\mu\nu}{}_\sigma \quad \rightarrow \quad T^{\mu\nu}{}_\nu$ vector

    $$T'^{\mu\nu}{}_\nu = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \widetilde{\Lambda}_\nu{}^\tau T^{\alpha\beta}{}_\tau = \Lambda^\mu{}_\alpha \delta^\tau{}_\beta T^{\alpha\beta}{}_\tau = \Lambda^\mu{}_\alpha T^{\alpha\beta}{}_\beta$$

  - Gradients of Tensor Fields: A tensor field is a tensor that is a function of the spacetime coordinates in Cartesian coordinate systems. The gradient operation $\partial/\partial x^\mu \equiv \partial_\mu$ acting on such a field produces a tensor field of on higher rank with μ as a new covariant index.

$$\lambda \quad \rightarrow \quad \frac{\partial \lambda}{\partial x^\mu} \equiv \partial_\mu \lambda \equiv \lambda,_\mu \quad \text{vector (gradient)} \qquad A^\mu \quad \rightarrow \quad \frac{\partial A^\mu}{\partial x^\mu} \equiv \partial_\mu A^\mu \equiv A^\mu,_\mu \quad \text{scalar (divergence)}$$

  - **Invariance of form or Lorentz covariance or covariance**: A fundamental property of a tensor equation is that if it is true in one Lorentz frame, then it is true in all Lorentz frames. Covariance plays a powerful role in helping decide what the proper equations of physics are.