# Radiative Processes in Astrophysics

## Lecture 4
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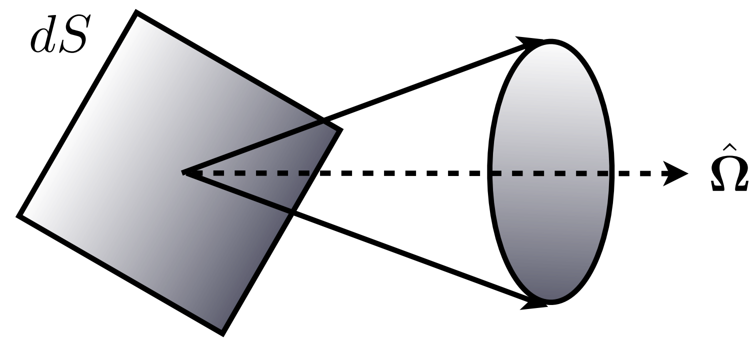
# More about Intensity, Flux

# Intensity and Flux

- Consider a ray propagating in the direction $\hat{\mathbf{\Omega}}$ and an area element $dS$ normal to $\hat{\mathbf{\Omega}}$.




- Intensity is the energy crossing a unit area normal to the ray direction per unit time per unit frequency per solid angle.

$$I_\nu = \frac{dE_\nu}{dS dt d\Omega d\nu}$$

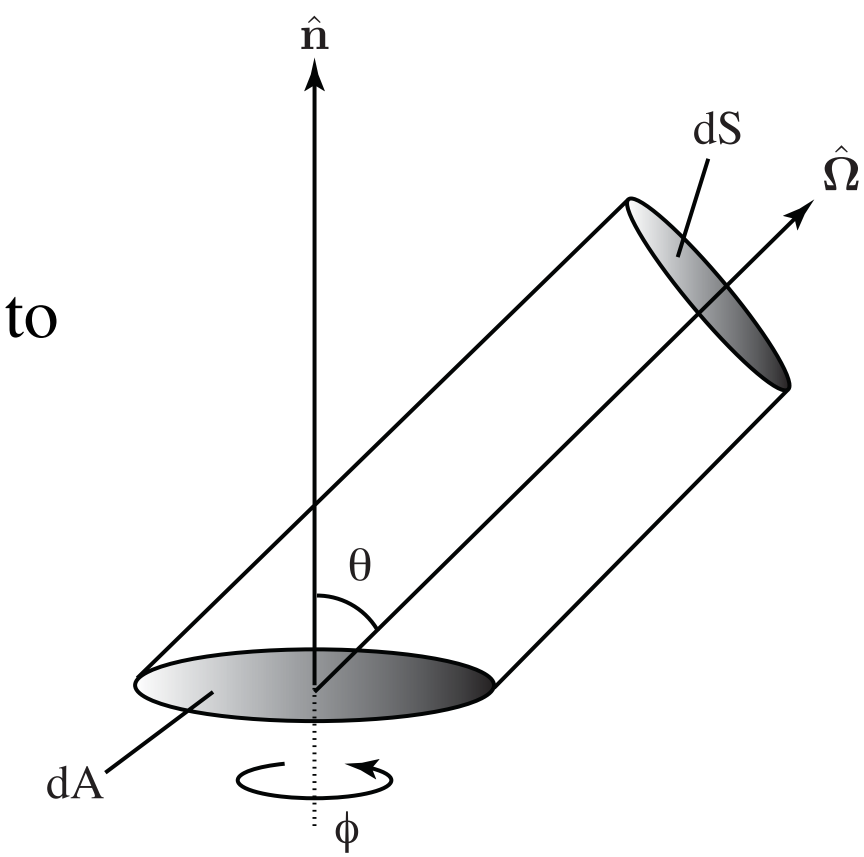
- Flux is the energy crossing a unit area normal to a given orientation $\hat{\mathbf{n}}$ per unit time per unit frequency, integrated over all solid angle.

  The contribution of the ray with a propagation direction $\hat{\mathbf{\Omega}}$ to the flux is

$$dF_\nu = \frac{dE_\nu(\Omega)}{dA dt d\Omega d\nu} d\Omega = \frac{I_\nu dS dt d\Omega d\nu}{dA dt d\Omega d\nu} d\Omega = I_\nu \cos\theta d\Omega$$

  Then, the flux is given by integrating over all solid angle

$$F_\nu = \int I_\nu \cos\theta d\Omega$$




$$dS = dA\cos\theta$$

# Constancy of Specific Intensity in Free Space

- Consider a bundle of rays and two points along the rays.

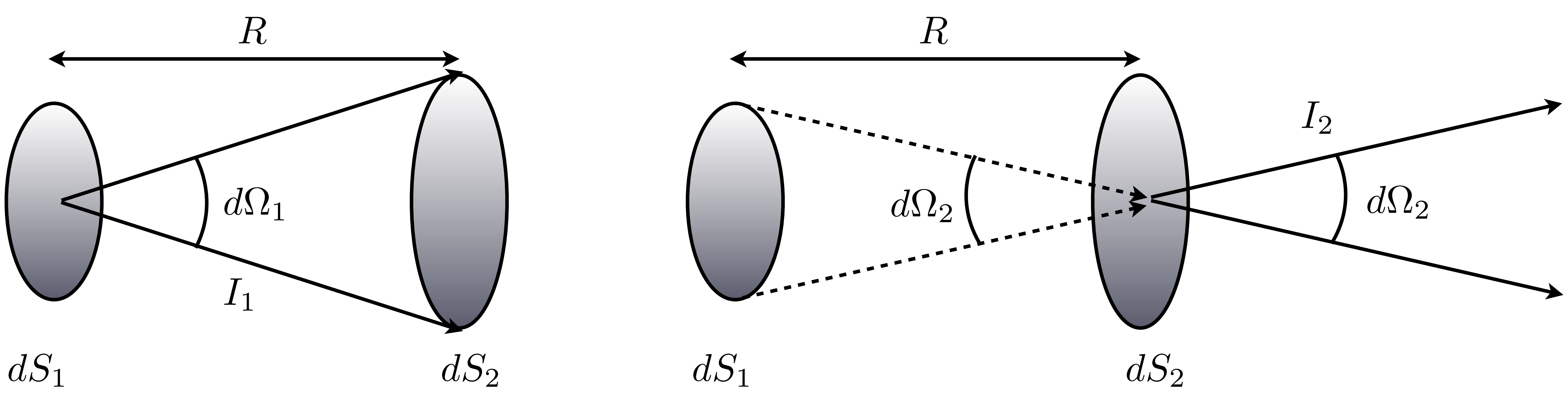
  Energy going out of "point 1" toward "point 2": $dE_1 = I_1 dS_1 dt d\Omega_1 d\nu$

  Energy coming into "point 2" from "point 1": $dE_2 = I_2 dS_2 dt d\Omega_2 d\nu$

- Because energy is conserved:

$$dE_1 = dE_2$$
$$I_1 dS_1 d\Omega_1 dt d\nu = I_2 dS_2 d\Omega_2 dt d\nu$$

Since $d\Omega_1 = dS_2/R^2$, $d\Omega_2 = dS_1/R^2$ , we obtain $I_1 = I_2$ .



# Pressure flux

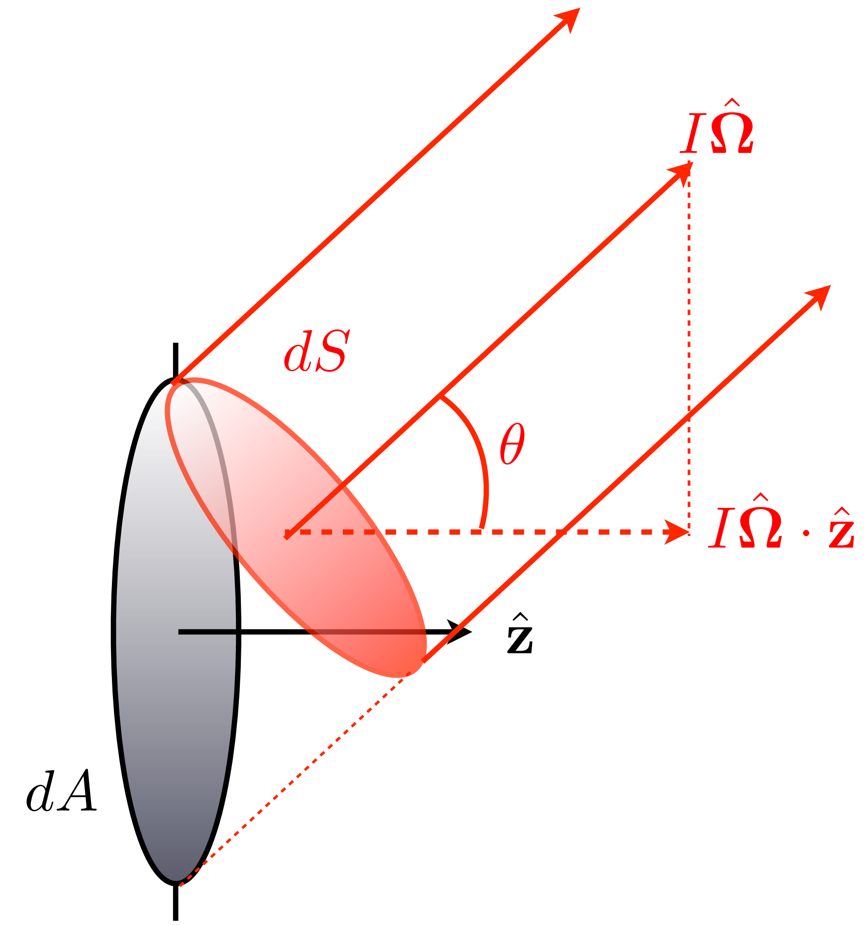
- Consider the $z$-component of the energy flow passing through the area $dA$ normal to $\hat{\mathbf{z}}$.

  area normal to $\hat{\mathbf{z}}$ : $dA = \dfrac{dS}{\cos\theta}$

  $z$-component of the energy flow = $(I\hat{\mathbf{\Omega}}) \cdot \hat{\mathbf{z}}\ d\Omega dS dt d\nu = I\cos\theta\ d\Omega(dA\cos\theta)dtd\nu$

- Pressure flux = the flux of $z$-component of the energy flow passing through $dA$ :

$$p_\nu = \frac{1}{c}\int I_\nu \cos^2\theta d\Omega$$



# Radiation from Moving Charges

# Potentials of a single moving charge (the Lienard-Wiechart potentials)

- Retarded potentials:

$$\phi(\mathbf{r},t) = \int d^3\mathbf{r}' \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}, \qquad \mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int d^3\mathbf{r}' \frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}$$

$$t' \equiv t_{\text{ret}} = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}$$

- It is convenient to rewrite the equations as integrals over 4D spacetime:

$$\phi(\mathbf{r},t) = \int d^3\mathbf{r}' \int dt' \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int d^3\mathbf{r}' \int dt' \frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c)$$

- Consider a particle of charge $q$ that moves along a trajectory $\mathbf{r} = \mathbf{r}_0(t)$. Its velocity is then $\mathbf{u}(t) = \dot{\mathbf{r}}_0(t)$ . The charge and current densities are given by

$$\rho(\mathbf{r},t) = q\delta(\mathbf{r}-\mathbf{r}_0(t)), \qquad \mathbf{j}(\mathbf{r},t) = q\mathbf{u}(t)\delta(\mathbf{r}-\mathbf{r}_0(t))$$

- Then, the potentials become

$$\phi(\mathbf{r},t) = q\int \frac{dt'}{|\mathbf{r}-\mathbf{r}_0(t')|}\delta(t'-t+|\mathbf{r}-\mathbf{r}_0(t')|/c)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{c}\int \frac{dt'\,\mathbf{u}(t')}{|\mathbf{r}-\mathbf{r}_0(t')|}\delta(t'-t+|\mathbf{r}-\mathbf{r}_0(t')|/c)$$

# Note on the Dirac delta function.

- $$\delta(\mathbf{r}-\mathbf{r}_0) = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$$

- $$\int f(x)\delta(x-x_0)dx = f(x_0)$$ if $x_0$ is not a function of $x$.

- $$\int f(x)\delta(g(x))dx = \int f(x)\delta(y)\frac{dy}{(dg/dx)} = \sum_{x_j} \frac{f(x_j)}{dg/dx|_{x_j}}$$

$$\leftarrow \quad y \equiv g(x'), \quad dy = (dg/dx')dx', \quad dx' = \frac{dy}{(dg/dx')}$$

where $x_j$ are roots of the equation $y = g(x) = 0$

- Let's define

$$\mathbf{R}(t') \equiv \mathbf{r} - \mathbf{r}_0(t') \quad \rightarrow \quad R(t') = |\mathbf{r} - \mathbf{r}_0(t')|$$

- We then have

$$\phi(\mathbf{r}, t) = q \int \frac{dt'}{R(t')} \delta(t' - t + R(t')/c)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{q}{c} \int \frac{\mathbf{u}(t')dt'}{R(t')} \delta(t' - t + R(t')/c)$$

- Change of variables:

$$t'' = t' - t + R(t')/c \quad \rightarrow \quad dt'' = \left[1 + \frac{1}{c}\dot{R}(t')\right] dt'$$

$$R^2(t') = \mathbf{R}(t') \cdot \mathbf{R}(t')$$

$$2R(t')\dot{R}(t') = -2\mathbf{R}(t') \cdot \mathbf{u}(t') \qquad \leftarrow \dot{\mathbf{R}}(t') = -\mathbf{u}(t')$$

$$\dot{R}(t') = -\frac{\mathbf{R}(t')}{R(t')} \cdot \mathbf{u}(t')$$

$$\mathbf{n}(t') \equiv \frac{\mathbf{R}(t')}{R(t')} \quad \rightarrow \quad \dot{R}(t') = -\mathbf{n}(t') \cdot \mathbf{u}(t')$$

$$dt'' = \left[1 - \frac{1}{c}\mathbf{n}(t') \cdot \mathbf{u}(t')\right] dt'$$

$$\kappa(t') \equiv 1 - \frac{1}{c}\mathbf{n}(t') \cdot \mathbf{u}(t') \quad \rightarrow \quad dt'' = \kappa(t')dt'$$

# The Lienard-Wiechart Potential

$$\phi(\mathbf{r},t) = q\int \frac{dt''}{\kappa(t')R(t')}\delta(t'')$$
$$\mathbf{A}(\mathbf{r},t) = \frac{q}{c}\int \frac{dt''\mathbf{u}(t')}{\kappa(t')R(t')}\delta(t'')$$

The equations then becomes by setting $t'' = 0 \quad \rightarrow \quad t' = t_{\mathrm{ret}} \equiv t - R(t_{\mathrm{ret}})/c$

$$\phi(\mathbf{r},t) = \frac{q}{\kappa(t_{\mathrm{ret}})R(t_{\mathrm{ret}})}$$
$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\frac{q\mathbf{u}(t_{\mathrm{ret}})}{\kappa(t_{\mathrm{ret}})R(t_{\mathrm{ret}})}$$

$Li\acute{e}nard - Wiechart\ potentials$
(리에나르-비헤르트)
(French-German)

- **Beaming effect**: The factor $\kappa(t_{\mathrm{ret}}) = 1 - \mathbf{n}(t_{\mathrm{ret}})\cdot\mathbf{u}(t_{\mathrm{ret}})/c$ becomes very important at velocities close to the speed of light, where, it tends to concentrate the potentials into a narrow cone about the particle velocity.
- **Retardation makes it possible for a particle to radiate**: In static case, differentiation of the $1/r$ potential to find the fields gives a $1/r^2$ decrease. However, the implicit dependence of the retarded time on position gives $1/r$ behavior of the fields. This allows radiation energy to flow to infinite distances.

# Electromagnetic Fields

- Then, the electromagnetic fields is obtained (see Jackson 14.1):
- Note $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{n}$ form a right-handed triad of mutually perpendicular vectors, and that $|\mathbf{E}| = |\mathbf{B}|$ . These properties are consistent with the solutions of the source-free Maxwell equations.

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

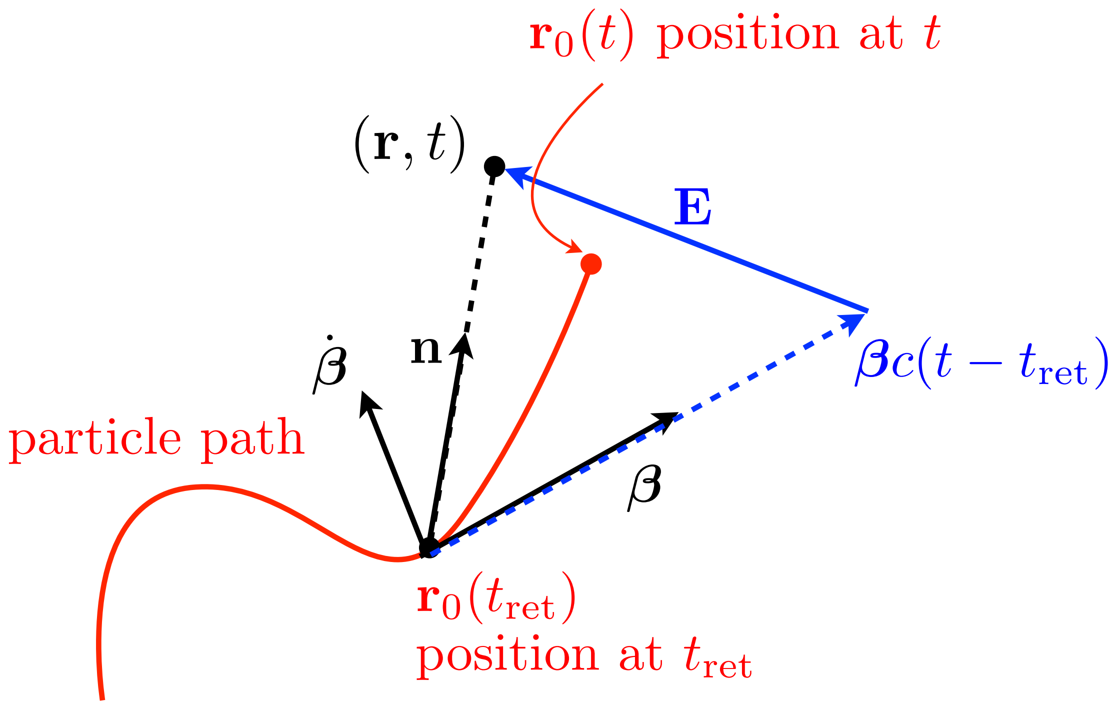
$$\mathbf{B} = \nabla \times \mathbf{A}$$

$\longrightarrow$

velocity field — acceleration field

$$\mathbf{E}(\mathbf{r}, t) = q\left[\frac{(\mathbf{n} - \boldsymbol{\beta})(1-\beta^2)}{\kappa^3 R^2}\right] + \frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 R} \times \{(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\}\right]$$

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{n} \times \mathbf{E}(\mathbf{r}, t)$$




Geometry for calculation of the radiation field at a point $(\mathbf{r}, t)$ in spacetime.

where

$$\mathbf{u} \equiv \dot{\mathbf{r}}_0(t_{\text{ret}})$$

$$\boldsymbol{\beta} \equiv \frac{\mathbf{u}(t_{\text{ret}})}{c} = \frac{\dot{\mathbf{r}}_0(t_{\text{ret}})}{c}$$

$$\dot{\boldsymbol{\beta}} \equiv \frac{\dot{\mathbf{u}}(t_{\text{ret}})}{c} = \frac{\ddot{\mathbf{r}}_0(t_{\text{ret}})}{c}$$

$$\mathbf{R} \equiv \mathbf{r} - \mathbf{r}_0(t_{\text{ret}})$$

$$\mathbf{n} \equiv \frac{\mathbf{R}}{R} = \frac{\mathbf{r} - \mathbf{r}_0(t_{\text{ret}})}{|\mathbf{r} - \mathbf{r}_0(t_{\text{ret}})|}$$

$$\kappa \equiv 1 - \mathbf{n} \cdot \boldsymbol{\beta}$$

# "Velocity" Field

- The first term depends only on position and velocity. When the particle moves with constant velocity it is only this term that contributes to the fields.

$$\mathbf{E}_{\text{vel}}(\mathbf{r},t) = q\left[\frac{(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2)}{\kappa^3 R^2}\right]$$

Displacement (of the photon) from the retarded point $\mathbf{r}_0(t_{\text{ret}})$ (point at $t_{\text{ret}}$) to the field point $\mathbf{r}$ during the light travel time = $\mathbf{n}c(t-t_{\text{ret}})$ .

In the same time, the particle undergoes a displacement $\boldsymbol{\beta}c(t-t_{\text{ret}})$.

The displacement between the field point and the current position of the particle is $(\mathbf{n}-\boldsymbol{\beta})c(t-t_{\text{ret}})$, which is the direction of the velocity field.

- Therefore, **the "velocity" electric field always points along the line toward the "current" position of the particle.**

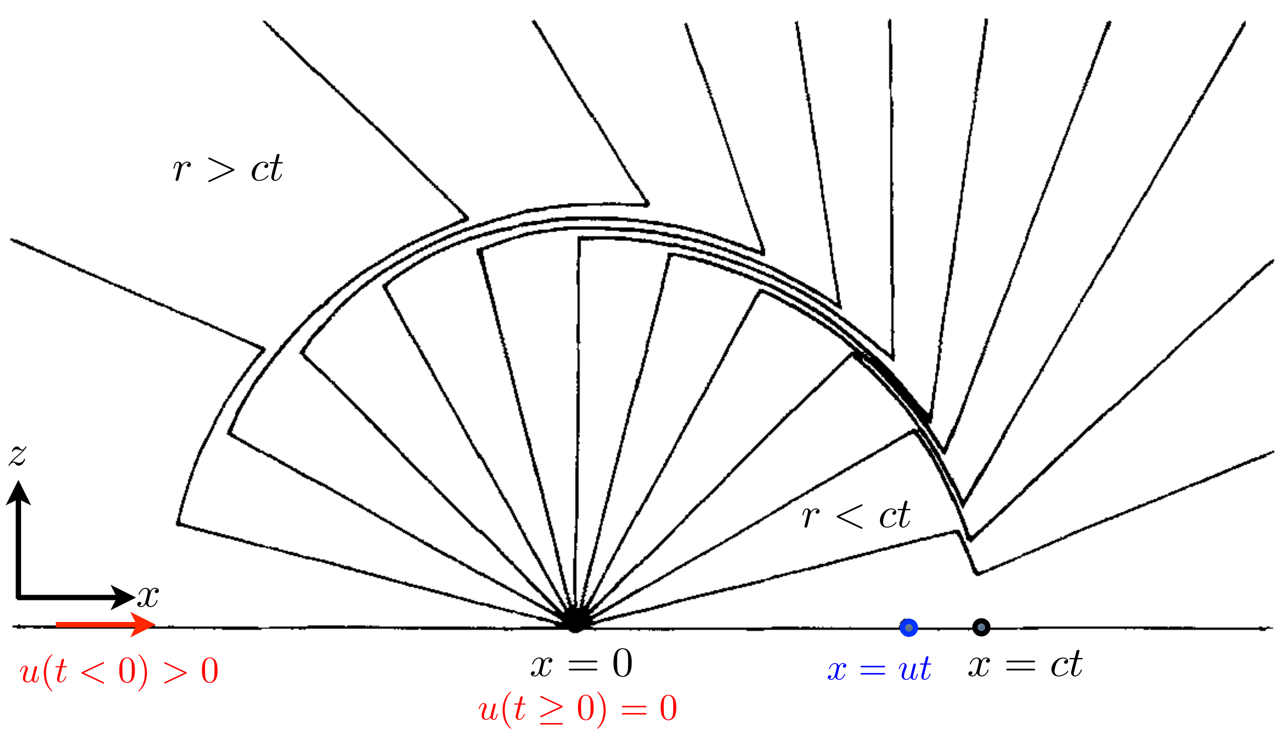
- The "velocity" field becomes precisely Coulomb's law as $u \ll c$ $(\beta \ll 1)$.

# "Acceleration" Field

- The second term (1) falls off as $1/R$, (2) is proportional to the particle's acceleration, and (3) is perpendicular to $\mathbf{n}$.

$$\mathbf{E}_{\rm rad}(\mathbf{r},t) = \frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 R}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\right]$$
$$\mathbf{B}_{\rm rad}(\mathbf{r},t) = \mathbf{n}\times\mathbf{E}_{\rm rad}(\mathbf{r},t)$$

- Let's consider a particle, which originally moved with a constant velocity along the $x$-axis and stopped at $x = 0$ at time $t = 0$. At time $t$, the field outside radius $ct$ is radial and points to the position where the particle would have been if there had been no deceleration, since no information of the deceleration has yet propagated. On the other hand, the field inside radius $ct$ is "informed" and is radially directed to the true position of the particle.




The figure demonstrates how an acceleration can give rise to a transverse field that decreases as $1/R$.
The radial thickness of transition zone will be constant. However, the radius of the zone increases as $R$. Since the total number of flux lines (in $xy$-plane) is conserved.

$$E\delta x(2\pi R) = \text{constant}$$
$$\therefore\ E \propto \frac{1}{R}$$

# Radiation Power

- Power per unit frequency per unit solid angle

$$
\begin{aligned}
\frac{dW}{d\omega d\Omega} &= \frac{R^2 dW}{d\omega dA} = R^2 c|\bar{\mathbf{E}}(\omega)|^2 \\
&= \frac{c}{4\pi^2}\left|\int [R\mathbf{E}(t)]\, e^{i\omega t} dt\right|^2 \\
&= \frac{q^2}{4\pi^2 c}\left|\int \left[\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\kappa^{-3}\right] e^{i\omega t} dt\right|^2
\end{aligned}
$$

Note: the expression in the brackets is evaluated at the retarded time $t' = t - R(t')/c$.

change of variables: $t' = t - R(t')/c, \quad dt = \kappa(t')dt' = (1 - \mathbf{n}(t')\cdot\boldsymbol{\beta}(t'))dt'$

$$
\frac{dW}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\int \mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\kappa^{-2} e^{i\omega(t'+R(t')/c)} dt'\right|^2
$$

If $|\mathbf{r}_0| \ll |\mathbf{r}| = r$, (1) $R(t') = |\mathbf{r}-\mathbf{r}_0(t')| = [(\mathbf{r}-\mathbf{r}_0)\cdot(\mathbf{r}-\mathbf{r}_0)]^{1/2}$

$$
\begin{aligned}
&= \left[r^2 - 2(\mathbf{r}\cdot\mathbf{r}_0) + r_0^2\right]^{1/2} = r\left[1 - \frac{2(\mathbf{r}\cdot\mathbf{r}_0)}{r^2} + \frac{r_0^2}{r^2}\right]^{1/2} \\
&\approx r\left(1 - \frac{\mathbf{r}\cdot\mathbf{r}_0}{r^2}\right) \\
&= r - \mathbf{n}\cdot\mathbf{r}_0 \qquad \leftarrow \quad \mathbf{n} \equiv \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|} \approx \frac{\mathbf{r}}{r}
\end{aligned}
$$

(2) $\kappa(t') = 1 - \mathbf{n}(t')\cdot\boldsymbol{\beta}(t') \approx 1 - \mathbf{n}\cdot\boldsymbol{\beta}(t')$ where $\mathbf{n}$ is independent of $t'$.

(3) $e^{i\omega(t'+R(t')/c)} = e^{i\omega r/c} e^{i\omega(t'-\mathbf{n}\cdot r_0(t')/c)}, \quad |e^{i\omega r/c}| = 1 \quad \rightarrow$

$$\therefore \ \frac{dW}{d\omega d\Omega} \approx \frac{q^2}{4\pi^2 c} \left| \int \mathbf{n} \times \{(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\} \kappa^{-2} e^{i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)} dt' \right|^2$$

We can integrate the above equation by parts to obtain an expression without $\dot{\boldsymbol{\beta}}$.

How?

We first need to show that: $\kappa^{-2}\mathbf{n} \times \{(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\} = \frac{d}{dt'}\left[\kappa^{-1}\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta})\right]$

With the rule: $\int f' g dt = fg - \int f g' dt$

we obtain

$$\int \mathbf{n} \times \{(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\} \kappa^{-2} e^{i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)} dt'$$

$$= \mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta})\kappa^{-1} e^{i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)}\Big|_{-\infty}^{\infty} - \int \mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta})\kappa^{-1}\{i\omega(1 - \mathbf{n}\cdot\dot{\mathbf{r}}_0(t')/c)\} e^{i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)} dt'$$

This term vanishes under the assumption of a finite wave train.

$$= -i\omega \int \mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta})\kappa^{-1}(1 - \mathbf{n}\cdot\boldsymbol{\beta}) e^{i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)} dt'$$

$$= -i\omega \int \mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta}) e^{i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)} dt' \qquad \leftarrow \quad \kappa = 1 - \mathbf{n}\cdot\boldsymbol{\beta}$$

$$\frac{dW}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2 c} \left| \int \mathbf{n} \times (\mathbf{n} \times \beta) \exp\left[ i\omega \left( t' - \frac{\mathbf{n}\cdot\mathbf{r}(t')}{c} \right) \right] dt' \right|^2$$

This formula will be used later.

- Proof of the relation:

$$\kappa^{-2}\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\} = \frac{d}{dt'}\left[\kappa^{-1}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\right]$$

note the vector identity: $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{C}\cdot\mathbf{A}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

$$\frac{d}{dt'}\left[\kappa^{-1}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\right] = \kappa^{-2}\left[-\frac{d\kappa}{dt'}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}) + \kappa\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})\right]$$

$$\text{Here, use the relations : } \kappa = 1-\mathbf{n}\cdot\boldsymbol{\beta}, \quad \frac{d\kappa}{dt'} = -\mathbf{n}\cdot\dot{\boldsymbol{\beta}}$$

$$\begin{aligned}
\frac{d}{dt'}\left[\kappa^{-1}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\right] &= \kappa^{-2}\left[(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\} + (1-\mathbf{n}\cdot\boldsymbol{\beta})\{\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})\}\right] \\
&= \kappa^{-2}\left[(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\} + \mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}}) - (\mathbf{n}\cdot\boldsymbol{\beta})\{\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})\}\right] \\
&= \kappa^{-2}\left[(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\{\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\beta}) - \boldsymbol{\beta}\} + \mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}}) - (\mathbf{n}\cdot\boldsymbol{\beta})\{\mathbf{n}(\mathbf{n}\cdot\dot{\boldsymbol{\beta}}) - \dot{\boldsymbol{\beta}}\}\right] \\
&= \kappa^{-2}\left[-(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\boldsymbol{\beta} + (\mathbf{n}\cdot\boldsymbol{\beta})\dot{\boldsymbol{\beta}} + \mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})\right] \\
&= \kappa^{-2}\left[-\mathbf{n}\times(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}) + \mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})\right] \\
&= \kappa^{-2}\left[\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\right]
\end{aligned}$$

# Radiation from Nonrelativistic Particles

- The previous formulae is fully relativistic. However, for the moment, we will discuss nonrelativistic particles:

$$\beta = \frac{u}{c} \ll 1$$

- Order of magnitude comparison of the two fields:

$$E_{\rm rad} \approx \frac{q}{c}\frac{\dot{\beta}}{\kappa^3 R}, \quad E_{\rm vel} \approx \frac{q}{\kappa^3 R^2} \quad \rightarrow \quad \frac{E_{\rm rad}}{E_{\rm vel}} \approx \frac{R\dot{u}}{c^2}$$

If the particle has a characteristic frequency of oscillation $\nu \sim 1/T$, then $\dot{u} \sim u\nu$, and the above equation becomes:

$$\frac{E_{\rm rad}}{E_{\rm vel}} \sim \frac{Ru\nu}{c^2} = \frac{u}{c}\frac{R}{\lambda}$$

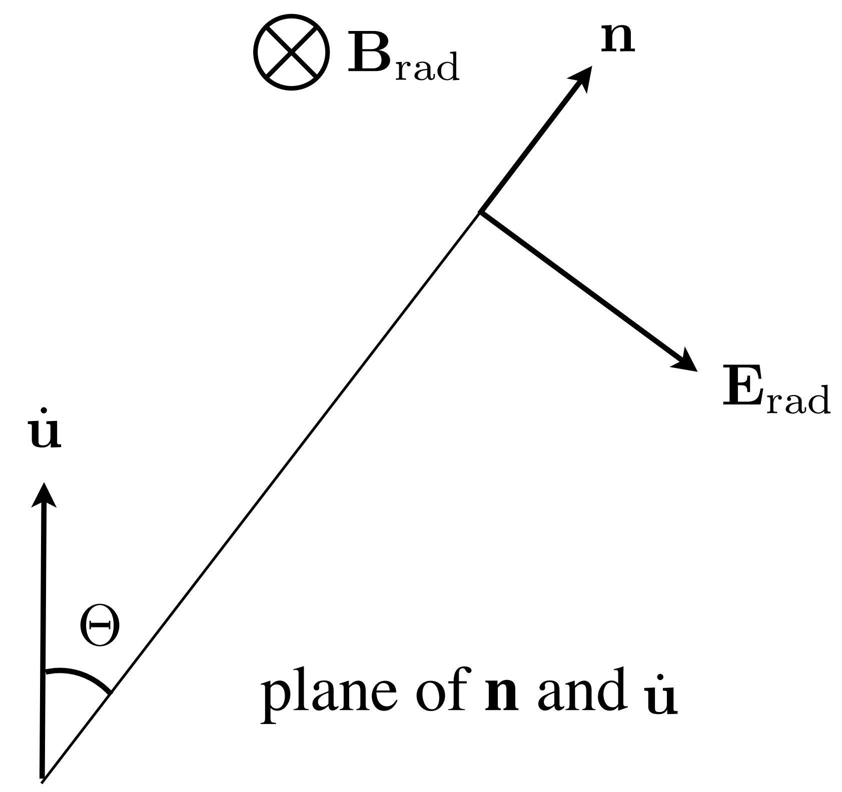
For field points inside the "near zone", $R \lesssim \lambda$, the velocity field is stronger than the radiation field by a factor $c/u = 1/\beta$.

For field points sufficiently far in the "far zone", $R \gg \lambda(c/u)$, the radiation field dominates.

# Larmor's Formula

- When $\beta \ll 1$,

$$\mathbf{E}_{\rm rad}(\mathbf{r},t) = \frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 R}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\right]$$
$$\approx \left[\frac{q}{Rc^2}\mathbf{n}\times(\mathbf{n}\times\dot{\mathbf{u}})\right]$$
$$\mathbf{B}_{\rm rad}(\mathbf{r},t) = \mathbf{n}\times\mathbf{E}_{\rm rad}(\mathbf{r},t)$$




Note:

$$\mathbf{n}\times(\mathbf{n}\times\dot{\mathbf{u}}) = \mathbf{n}(\mathbf{n}\cdot\dot{\mathbf{u}}) - \dot{\mathbf{u}}$$
$$\{\mathbf{n}\times(\mathbf{n}\times\dot{\mathbf{u}})\}^2 = (\mathbf{n}\cdot\dot{\mathbf{u}})^2 + (\dot{\mathbf{u}})^2 - 2(\mathbf{n}\cdot\dot{\mathbf{u}})^2$$
$$= \dot{u}^2\cos^2\Theta + \dot{u}^2 - 2\dot{u}^2\cos^2\Theta$$
$$= \dot{u}^2(1-\cos^2\Theta)$$
$$= \dot{u}^2\sin^2\Theta$$

$$\therefore\ |\mathbf{E}_{\rm rad}| = |\mathbf{B}_{\rm rad}| = \frac{q\dot{u}}{Rc^2}\sin\Theta$$

- The Poynting vector is in direction of **n** and has a magnitude.

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} = \frac{c}{4\pi}E_{\rm rad}^2\mathbf{n}$$
$$S = \frac{c}{4\pi}\frac{q^2\dot{u}^2}{R^2c^4}\sin^2\Theta \equiv \frac{dW}{dtdA} \quad ({\rm erg\ s^{-1}\ cm^{-2}})$$

- Energy emitted per unit time into solid angle about **n**:

$$\frac{dW}{dtd\Omega} = R^2 \frac{dW}{dtdA} = R^2 S$$
$$= \frac{q^2 \dot{u}^2}{4\pi c^3} \sin^2 \Theta$$

$$\boxed{\frac{dP}{d\Omega} = \frac{q^2 \dot{u}^2}{4\pi c^3} \sin^2 \Theta}$$

- Total power emitted into all angles:

$$P = \frac{dW}{dt} = \int d\Omega \frac{q^2 \dot{u}^2}{4\pi c^3} \sin^2 \Theta = \frac{q^2 \dot{u}^2}{2c^3} \int_{-1}^{1} (1 - \mu^2) d\mu$$

$$\boxed{P = \frac{2q^2 \dot{u}^2}{3c^3}}$$

Larmor's Formula

1. The Power emitted is proportional to the square of the charge and the square of the acceleration.
2. Characteristic dipole pattern $\propto \sin^2 \Theta$ : no radiation is emitted along the direction of acceleration, and the maximum is emitted perpendicular to acceleration.
3. The instantaneous direction of $\mathbf{E}_{\rm rad}$ is determined by $\dot{\mathbf{u}}$ and **n**. If the particle accelerates along a line, the radiation will be 100% linearly polarized in the plane of $\dot{\mathbf{u}}$ and **n**.

# Dipole Approximation

- Consider many particles with positions $\mathbf{r}_i$, velocities $\mathbf{u}_i$, and charges $q_i (i = 1, 2, \cdots N)$. The radiation field at large distances can be found by adding together the $\mathbf{E}_{\rm rad}$ from each particle.
- However, the radiation field equations refer to conditions at retarded time, and the retarded times will differ for each particle. Therefore, we must keep track of the phase relations between the particles.
- There are situations in which it is possible to ignore this difficulty:

Let $L$ = typical size of the system
$\tau$ = typical time scale for variations within the system

If $\tau \gg L/c$ (light-travel-time), the differences in retarded time across the source are negligible.

Note that $\tau$ can represent the time scale over which significant changes in the radiation field, and this in turn determines typical characteristic frequency of the emitted radiation.

The above condition is equivalent to the condition for the characteristic frequency (or characteristic wavelength) :

$$\nu \approx \frac{1}{\tau} \ll \frac{c}{L} \quad \text{or} \quad \lambda = \frac{c}{\nu} \gg L$$

Therefore, the differences in retarded times can be ignored when the system size is smaller than the characteristic wavelength.

We may also characterize $\tau$ as the time a particle takes to change its motion substantially.

Let $l$ be a characteristic scale of the particle's orbit and $u$ be a typical velocity,then $\tau \sim l/u$ .

The above condition $\tau \gg L/c$ then imply

$$u/c = l/(\tau c) \ll l/L$$

But since $l < L$ , this is simply equivalent to the nonrelativistic condition:

$$u \ll c$$

- With the above conditions met we can write:

$$\mathbf{E}_{\rm rad} = \sum_i \frac{q_i}{c^2} \frac{\mathbf{n} \times (\mathbf{n} \times \dot{\mathbf{u}}_i)}{R_i}$$

- Let $R_0$ be the distance from some point in the system to the field point.

$$R_i = R_0 + l_i \approx R_0 \text{ as } R_0 \to \infty$$

- Then, we have

$$\mathbf{E}_{\rm rad} = \frac{1}{c^2} \frac{\mathbf{n} \times (\mathbf{n} \times \sum_i q_i \dot{\mathbf{u}}_i)}{R_0}$$

$$= \frac{\mathbf{n} \times (\mathbf{n} \times \ddot{\mathbf{d}})}{c^2 R_0}$$

where the dipole moment

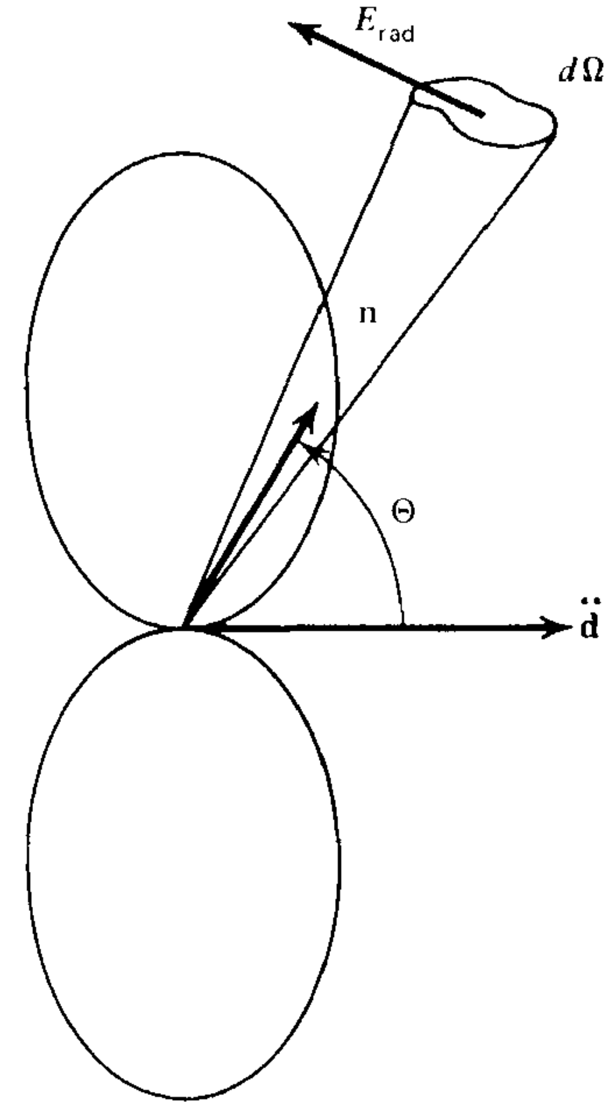
$$\ddot{\mathbf{d}} \equiv \sum_i q_i \mathbf{r}_i$$

Note that the right-hand side of the above equations must still be evaluated at a retarded time, but using any point within the region, say, $R_0$ .

- Dipole approximation:

$$\frac{dP}{d\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3}\sin^2\Theta, \quad P = \frac{2\ddot{\mathbf{d}}^2}{3c^3}$$

The instantaneous polarization of E lines in the plane of $\ddot{\mathrm{d}}$ and **n**.




- Power spectrum of radiation in the dipole approximation:

For simplicity we assume that d always lies in a single direction.

$$E(t) = \ddot{d}(t)\frac{\sin\Theta}{c^2 R_0}, \quad d(t) = \int_{-\infty}^{\infty} e^{-i\omega t}\bar{d}(\omega)d\omega$$

$$\ddot{d}(t) = -\int_{-\infty}^{\infty} \omega^2 e^{-i\omega t}\bar{d}(\omega)d\omega$$

$$\therefore\ \bar{E}(\omega) = -\frac{1}{c^2 R_0}\omega^2\bar{d}(\omega)\sin\Theta$$

$$\frac{dW}{d\omega d\Omega} = R_0^2\frac{dW}{d\omega dA} \quad \rightarrow \quad \begin{aligned} \frac{dW}{d\omega d\Omega} &= \frac{1}{c^3}\omega^4|\bar{d}(\omega)|^2\sin^2\Theta \\ \frac{dW}{d\omega} &= \frac{8\pi\omega^4}{3c^3}|\bar{d}(\omega)|^2 \end{aligned}$$

Note the $\omega^4 \propto \lambda^{-4}$ dependence.

the spectrum of the emitted radiation is related directly to the frequencies of oscillation of the dipole moment. However, this property is not true for relativistic particles.

# Multipole Expansion

- Vector potential:

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int d^3\mathbf{r}' \int dt' \frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c)$$

- Consider a Fourier analysis of the sources and fields:

$$\mathbf{j}_\omega(\mathbf{r}) = \int \mathbf{j}(\mathbf{r},t)e^{i\omega t}dt$$

$$\mathbf{A}_\omega(\mathbf{r}) = \int \mathbf{A}(\mathbf{r},t)e^{i\omega t}dt = \frac{1}{c}\int d^3\mathbf{r}' \int dt' \int dt \frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}e^{i\omega t}\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c)$$

$$= \frac{1}{c}\int d^3\mathbf{r}' \int dt' \frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}e^{i\omega t'}e^{i\omega|\mathbf{r}-\mathbf{r}'|/c}$$

$$= \frac{1}{c}\int d^3\mathbf{r}' \frac{\mathbf{j}_\omega(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}e^{ik|\mathbf{r}-\mathbf{r}'|} \qquad \leftarrow \quad k \equiv \omega/c$$

Note this equation relate single Fourier components of source and potential.

- Let's choose an origin of coordinates inside the source of size $L$. At field points such that $r \ll L$.

$$|\mathbf{r}-\mathbf{r}'| = [(\mathbf{r}-\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')]^{1/2}$$

$$= \left[r^2 - 2(\mathbf{r}\cdot\mathbf{r}') + r'^2\right]^{1/2} = r\left[1 - \frac{2(\mathbf{r}\cdot\mathbf{r}')}{r^2} + \frac{r'^2}{r^2}\right]^{1/2}$$

$$\approx r\left(1 - \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right)$$

$$= r - \mathbf{n}\cdot\mathbf{r}' \qquad \leftarrow \quad \mathbf{n} \equiv \frac{\mathbf{r}}{r}$$

- The Fourier component becomes:

$$\mathbf{A}_\omega(\mathbf{r}) \approx (e^{ikr}/cr)\int \mathbf{j}_\omega(\mathbf{r}')e^{-ik\mathbf{n}\cdot\mathbf{r}'}d^3\mathbf{r}'$$

- The factor $\exp(ikr)$ outside the integral expresses the effect of retardation from the source as a whole. The factor $\exp(-ik\mathbf{n}\cdot\mathbf{r}')$ inside the integral expresses the relative retardation of each element of the source.

- In our slow-motion approximation, $kL = 2\pi L/\lambda \ll 1$. Expanding the exponential in the integral:

$$\mathbf{A}_\omega(\mathbf{r}) = \frac{e^{ikr}}{cr}\int \mathbf{j}_\omega(\mathbf{r}')\sum_{n=0}^{\infty}\frac{(-ik\mathbf{n}\cdot\mathbf{r}')^n}{n!}d^3\mathbf{r}'$$

Dipole approximation results from taking just the first term ($n = 0$):

$$\mathbf{A}_\omega(\mathbf{r})|_{\rm dipole} = \frac{e^{ikr}}{cr}\int \mathbf{j}_\omega(\mathbf{r}')d^3\mathbf{r}'$$

Quadrupole term is the second term:

$$\mathbf{A}_\omega(\mathbf{r})|_{\rm quad} = \frac{-ike^{ikr}}{cr}\int \mathbf{j}_\omega(\mathbf{r}')(\mathbf{n}\cdot\mathbf{r}')d^3\mathbf{r}'$$