Radiative Processes in Astrophysics

Lecture 3

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Useful Mathematical Formulae

- Dirac delta function:

$$\delta(\omega-\omega') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega-\omega')t}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(\omega-\omega')t}dt$$

- Fourier Transform:

Rybicki

$$\bar{a}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(t)e^{-i\omega t}dt$$

$$a(t) = \int_{-\infty}^{\infty} \bar{a}(\omega)e^{i\omega t}d\omega$$

Afken (Mathematical Methods for Physicists)

$$\bar{a}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} a(t)e^{i\omega t}dt$$

$$a(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{a}(\omega)e^{-i\omega t}d\omega$$

Parseval's Theorem

Rybicki:

$$\int_{-\infty}^{\infty} a(t)b^*(t)dt = 2\pi\int_{-\infty}^{\infty} \bar{a}(\omega)\bar{b}^*(\omega)d\omega$$

$$\int_{-\infty}^{\infty} |a(t)|^2dt = 2\pi\int_{-\infty}^{\infty} |\bar{a}(\omega)|^2d\omega$$

Afken:

$$\int_{-\infty}^{\infty} a(t)b^*(t)dt = \int_{-\infty}^{\infty} \bar{a}(\omega)\bar{b}^*(\omega)d\omega$$

$$\int_{-\infty}^{\infty} |a(t)|^2dt = \int_{-\infty}^{\infty} |\bar{a}(\omega)|^2d\omega$$

- Vector identities:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

$$\nabla\times(\nabla\phi) = 0$$

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0$$

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Basic Theory of Radiation Fields

Electromagnetic force on a single charged particle

- **Lorentz force**: If a particle of charge q and mass m moves with velocity $\mathbf{v}$ in the presence of an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$, then it will experience a force:

$$\boxed{\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)}$$

(in Gaussian units, or cgs units)

- **Power** supplied by the EM fields (the rate of work done by the fields) on a particle is

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{F} &= q\mathbf{v}\cdot\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \\
\mathbf{v}\cdot m\frac{d\mathbf{v}}{dt} &= q\mathbf{v}\cdot\mathbf{E} \\
\therefore \frac{dU_{\rm mech}}{dt} &\equiv \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = q\mathbf{v}\cdot\mathbf{E}
\end{aligned}$$

- Note $\mathbf{v}\cdot(\mathbf{v}\times\mathbf{B}) = 0$, meaning that the magnetic fields do not work.

Electromagnetic force on a continuous medium

- Consider a medium with **charge density** and **current density**:

$$\rho \equiv \lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_i q_i$$

$$\mathbf{j} \equiv \lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_i q_i \mathbf{v}_i$$

- **Force density** (force per unit volume):

$$\mathbf{f} = \rho \mathbf{E} + \frac{1}{c} \mathbf{j} \times \mathbf{B}$$

- **Power density** supplied by the field (the rate of work done by the field per unit volume):

$$\frac{du_{\text{mech}}}{dt} \equiv \lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_i q_i \mathbf{v}_i \cdot \mathbf{E} = \mathbf{j} \cdot \mathbf{E}$$

Note typos in the textbook. They use the same symbol to denote the energy density *u* and the total energy *U* within in a volume.

Maxwell's equations

- **Maxwell's eqs.** (in macroscopic forms) relates fields to charge and current densities.

$\nabla \cdot \mathbf{D} = 4\pi\rho$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	Gauss's law for magnetism (no magnetic monopoles)
$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$	Maxwell-Faraday equation
$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$	Ampere-Maxwell equation

D,**H** : macroscopic fields
B,**E** : microscopic fields

$$\mathbf{D} = \epsilon \mathbf{E}$$
$$\mathbf{B} = \mu \mathbf{H}$$

ϵ : dielectric constant

μ : magnetic permeability

Dielectric material (절연체): an electrical insulator that can be polarized by an applied electric field. Electric charges do not flow through the material as they do in a conductor, but only slightly shift from their average equilibrium positions causing dielectric polarization.

Permeability (투자율): the degree of magnetization of a material in response to a magnetic field.

Note $\epsilon = \mu = 1$ in the absence of dielectric or permeability media.

- **Conservation of charge**

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \left(\frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}\right) \quad \rightarrow \quad \boxed{\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0}$$

$$0 = \frac{4\pi}{c}\nabla \cdot \mathbf{j} + \frac{1}{c}\frac{\partial}{\partial t}(4\pi\rho)$$

Electromagnetic Field Energy

- Use the Ampere's law to obtain the mechanical energy density

$$\frac{du_{\rm mech}}{dt} = \mathbf{j}\cdot\mathbf{E} = \frac{1}{4\pi}\mathbf{E}\cdot\left(c\nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t}\right)$$

- Use a vector identity and Faraday's law:

$$\begin{aligned}\mathbf{E}\cdot(\nabla\times\mathbf{H}) &= \mathbf{H}\cdot(\nabla\times\mathbf{E}) - \nabla\cdot(\mathbf{E}\times\mathbf{H})\\ &= -\frac{1}{c}\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} - \nabla\cdot(\mathbf{E}\times\mathbf{H})\end{aligned}$$

Then,

$$\mathbf{j}\cdot\mathbf{E} = \frac{1}{4\pi}\left(-c\nabla\cdot(\mathbf{E}\times\mathbf{H}) - \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} - \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}\right)$$

- **Poynting's theorem** in differential form.

$$\mathbf{j}\cdot\mathbf{E} + \frac{1}{8\pi}\frac{\partial}{\partial t}\left(\epsilon E^2 + \frac{B^2}{\mu}\right) = -\nabla\cdot\left(\frac{c}{4\pi}\mathbf{E}\times\mathbf{H}\right)$$

- **Electromagnetic field density** (field energy per unit volume) and **Poynting vector** (electromagnetic flux vector) are identified:

$$\boxed{u_{\rm field} = \frac{1}{8\pi}\left(\epsilon E^2 + \frac{B^2}{\mu}\right) = u_E + u_B \qquad\qquad \mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}}$$

- The Poynting's theorem becomes an expression of the **local conservation of energy**.

$$\frac{\partial}{\partial t}(u_{\rm mech} + u_{\rm field}) + \nabla \cdot \mathbf{S} = 0$$

- Integrating the equation over a volume element and using the divergence theorem, we obtain the **conservation of energy**:

$$\boxed{\frac{d}{dt}(U_{\rm mech} + U_{\rm field}) = -\int_\Sigma \mathbf{S} \cdot d\mathbf{A}}$$

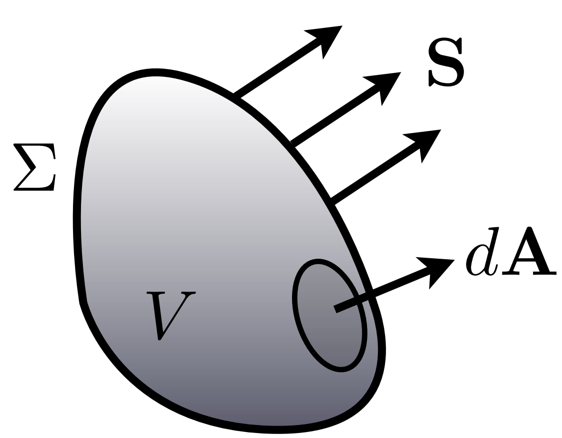
Here,

$$U_{\rm mech} \equiv \int_V u_{\rm mech} dV \quad \text{and} \quad U_{\rm field} \equiv \int_V u_{\rm field} dV$$

or

$$\int_V (\mathbf{j} \cdot \mathbf{E}) dV + \frac{d}{dt} \int_V \left(\frac{\epsilon E^2 + B^2/\mu}{8\pi} \right) dV = -\int_\Sigma \mathbf{S} \cdot d\mathbf{A}$$

Meaning: <u>the rate of change of total (mechanical + field) energy within the volume *V* is equal to the net inward flow of energy through the bounding surface Σ.</u>



divergence theorem:

$$\int_V \nabla \cdot \mathbf{S} dV = \int_\Sigma \mathbf{S} \cdot d\mathbf{A}$$

- In electrostatics and magnetostatics, we recall that

$$\mathbf{E} \propto r^{-2} \ \text{ and } \ \mathbf{B} \propto r^{-2} \ \text{ as } \ r \to \infty \qquad \to \quad \mathbf{S} \propto r^{-4}$$

$$\therefore \quad \int_\Sigma \mathbf{S} \cdot d\mathbf{A} = 0 \ \text{ as } \ r \to \infty$$

- However, for time-varying fields, we will find that

$$\mathbf{E} \propto r^{-1} \ \text{ and } \ \mathbf{B} \propto r^{-1} \ \text{ as } \ r \to \infty$$

$$\therefore \quad \int_\Sigma \mathbf{S} \cdot d\mathbf{A} \neq 0 \ \text{ as } \ r \to \infty$$

- This finite energy flowing outward (or inward) at large distances is called **radiation**. Those parts of **E** and **B** that decreases as r^{-1} at large distances are said to constitute the **radiation field**.

Electromagnetic Waves

- In vacuum $(\rho = 0 = \mathbf{j},\ \epsilon = 1 = \mu)$, Maxwell's equations give the vector wave equations:

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \qquad \nabla \times (\nabla \times \mathbf{B}) = \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \mathbf{E})$$

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \nabla(\nabla \cdot \mathbf{B}) - \nabla^2\mathbf{B} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}$$

$$\therefore\ \nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \qquad \therefore\ \nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

$$\longrightarrow \boxed{\begin{aligned} \nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} &= 0 \\ \nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} &= 0 \end{aligned}}$$

- Fourier transform of fields:

$$\bar{\mathbf{E}}(\mathbf{k},\omega) = \frac{1}{(2\pi)^4}\int d^3\mathbf{r}\int dt\mathbf{E}(\mathbf{r},t)e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

wavevector $\mathbf{k}$:

magnitude $k = 2\pi/\lambda = \omega/c$, direction $\hat{\mathbf{k}} = \mathbf{k}/k$

angular frequency : $\omega = 2\pi\nu$

- Inverse transformation:

$$\mathbf{E}(\mathbf{r},t) = \int d^3\mathbf{k}\int d\omega\bar{\mathbf{E}}(\mathbf{k},\omega)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

- Apply the wave equation to Fourier expansion:

$$\begin{aligned} \nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t^2} &= -\int d^3\mathbf{k}\int d\omega\left(k^2 - \frac{\omega^2}{c^2}\right)\mathbf{E}(\mathbf{k},\omega)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \\ &= 0 \end{aligned}$$

Dispersion relation

- We obtain the vacuum dispersion relation, phase velocity, and group velocity:

$$\omega = ck \qquad v_{\rm ph} \equiv \frac{\omega}{k} = c \qquad v_{\rm g} \equiv \frac{\partial\omega}{\partial k} = c$$

dispersion relation = a function which gives ω as a function of k.

phase velocity = the rate at which the phase of the wave propagates in space.

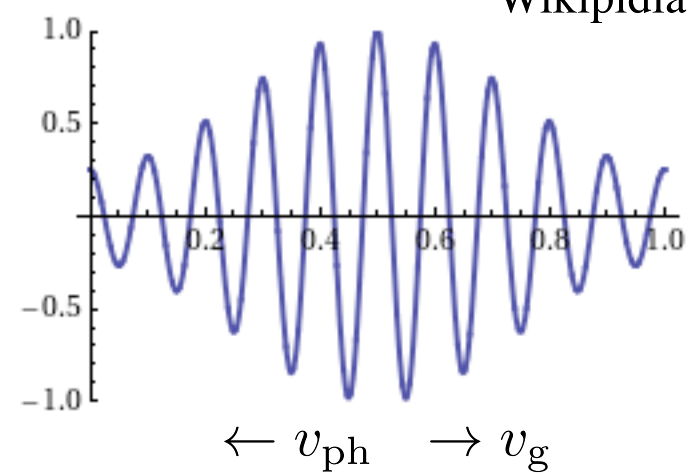
group velocity = the velocity with which the overall shape of the waves' amplitudes (modulation or envelope of the wave) propagates through space.

- Assume the wave packet E is almost monochromatic, so that its Fourier component is nonzero only in the vicinity of a central wavenumber k_0. Then, linearization gives:

$$\begin{aligned}\omega(k) &\approx \omega_0 + (k-k_0)\frac{\partial\omega(k)}{\partial k}|_{k=k_0} \\ &= \omega_0 + (k-k_0)\omega_0'\end{aligned}$$

$$\begin{aligned}E(x,t) &= \int dk \int d\omega \bar{E}(k,\omega) e^{i(kx-\omega t)} \\ &\approx e^{it(\omega_0' k_0 - \omega_0)} \int dk \bar{E}(k,\omega_0) e^{ik(x-\omega_0' t)}\end{aligned}$$

$$|E(x,t)| = |E(x-\omega_0' t, 0)|$$



- The envelope of the wavepacket travels at velocity $\omega_0' = (\partial\omega/\partial k)_{k=k_0}$.

Properties of a single Fourier mode

- Consider an arbitrary Fourier mode in vacuum: (E_0 and B_0 are complex constants.)

$$\mathbf{E} = E_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{a}}_1 \quad \mathbf{B} = B_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{a}}_2$$

- Substituting into Maxwell's equations yields:

$$\nabla\cdot\mathbf{E} = 0 \quad \rightarrow \quad \mathbf{k}\cdot\mathbf{E} = 0$$

$$\nabla\cdot\mathbf{B} = 0 \quad \rightarrow \quad \mathbf{k}\cdot\mathbf{B} = 0$$

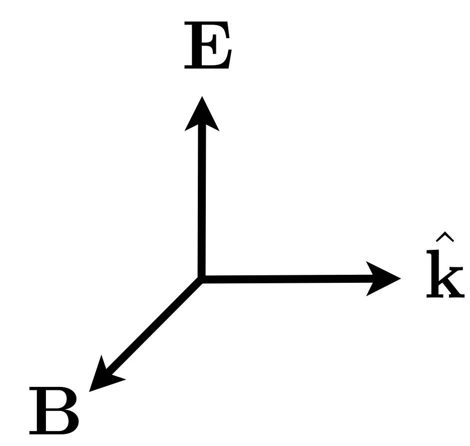
$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \quad \rightarrow \quad \mathbf{k}\times\mathbf{E} = \frac{\omega}{c}\mathbf{B}$$

$$\nabla\times\mathbf{B} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \quad \rightarrow \quad \mathbf{k}\times\mathbf{B} = -\frac{\omega}{c}\mathbf{E}$$

$$\left(\mathbf{k} = \hat{\mathbf{k}}\frac{\omega}{c}\right)$$

or $\hat{\mathbf{k}}\times\mathbf{E} = \mathbf{B}$

or $\hat{\mathbf{k}}\times\mathbf{B} = -\mathbf{E}$



$$\hat{\mathbf{k}}\times\mathbf{E} = \mathbf{B} \;\rightarrow\; \hat{\mathbf{k}}\times\hat{\mathbf{a}}_1 E_0 = \hat{\mathbf{a}}_2 B_0$$

$$\rightarrow\; E_0(\hat{\mathbf{k}}\times\hat{\mathbf{a}}_1)\cdot\hat{\mathbf{a}}_2 = B_0$$

$$\rightarrow\; E_0 = B_0$$

$$\begin{array}{l} E_0 = |\mathbf{E}|e^{i\phi_E} \\ B_0 = |\mathbf{B}|e^{i\phi_B} \end{array} \;\rightarrow\; \phi_E = \phi_B$$

(1) EM waves are transverse (perpendicular to the direction of propagation).

(2) **E** and **B** are orthogonal to each other.

(3) $(\hat{\mathbf{k}},\ \hat{\mathbf{a}}_1,\ \hat{\mathbf{a}}_2)$ form an orthogonal basis

(4) Field amplitudes are equal: $|\mathbf{B}| = |\mathbf{E}|,\quad B_0 = E_0$ and $\phi_B = \phi_E$

- If *A(t)* and *B(t)* are two complex quantities with the same sinusoidal time dependence,

$$A(t) = \mathcal{A}e^{i\omega t} \qquad B(t) = \mathcal{B}e^{i\omega t}$$

then the time average of the product of their real parts is

$$\begin{aligned}\langle \mathrm{Re}A(t)\cdot \mathrm{Re}B(t)\rangle &= \frac{1}{4}\left\langle \left(\mathcal{A}e^{i\omega t}+\mathcal{A}^*e^{-i\omega t}\right)\left(\mathcal{B}e^{i\omega t}+\mathcal{B}^*e^{-i\omega t}\right)\right\rangle \\ &= \frac{1}{4}\left\langle \mathcal{A}\mathcal{B}^* + \mathcal{A}^*\mathcal{B}\right\rangle \\ &= \frac{1}{2}\mathrm{Re}(\mathcal{A}\mathcal{B}^*) = \frac{1}{2}\mathrm{Re}(\mathcal{A}^*\mathcal{B})\end{aligned}$$

- Time-averaged Poynting vector amplitude:

note : $\mathbf{E}$ and $\mathbf{B}$ are real.

$$\begin{aligned}\langle S\rangle &= \frac{c}{4\pi}\left\langle \mathrm{Re}\left(E_0e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right)\mathrm{Re}\left(B_0e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right)\right\rangle \\ &= \frac{c}{8\pi}\mathrm{Re}\left(E_0B_0^*\right) \\ &= \frac{c}{8\pi}|E_0|^2 = \frac{c}{8\pi}|B_0|^2 \qquad \leftarrow E_0 = B_0\end{aligned}$$

- Time-averaged field energy density:

$$\langle U_{\mathrm{field}}\rangle = \frac{1}{8\pi}\left\langle |\mathbf{E}|^2 + |\mathbf{B}|^2\right\rangle = \frac{1}{16\pi}\mathrm{Re}(E_0E_0^* + B_0B_0^*) = \frac{1}{8\pi}|E_0|^2 = \frac{1}{8\pi}|B_0|^2$$

- Velocity of energy flow: $\langle S\rangle \,/\, \langle U_{\mathrm{field}}\rangle = c$

Power Spectrum

- A common property of any wave theory:

  If we have a time record of the radiation field of length Δt, we can only define the spectrum to within a frequency resolution $\Delta\omega$ where

  $$\boxed{\Delta\omega\Delta t > 1. \quad \text{(uncertainty relation)}}$$

- Let us consider only a component of the transverse electric field: $E(t) \equiv \hat{\mathbf{a}} \cdot \mathbf{E}(t)$
- Fourier transform and its inverse are:

$$\bar{E}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(t)e^{i\omega t}dt, \qquad E(t) = \int_{-\infty}^{\infty} \bar{E}(\omega)e^{-i\omega t}d\omega$$

  Since *E*(t) is real, the negative frequencies are redundant, i.e., $\bar{E}(-\omega) = \bar{E}^*(\omega)$.

- Total energy per unit area per unit time: $\dfrac{dW}{dAdt} = \dfrac{c}{4\pi}E^2(t)$ (Poynting vector)
- Total energy per unit area:

$$\frac{dW}{dA} = \int_{-\infty}^{\infty} \frac{dW}{dAdt}dt = \frac{c}{4\pi}\int_{-\infty}^{\infty} E^2(t)dt$$
$$= \frac{c}{2}\int_{-\infty}^{\infty} |\bar{E}(\omega)|^2 d\omega = c\int_{0}^{\infty} |\bar{E}(\omega)|^2 d\omega$$

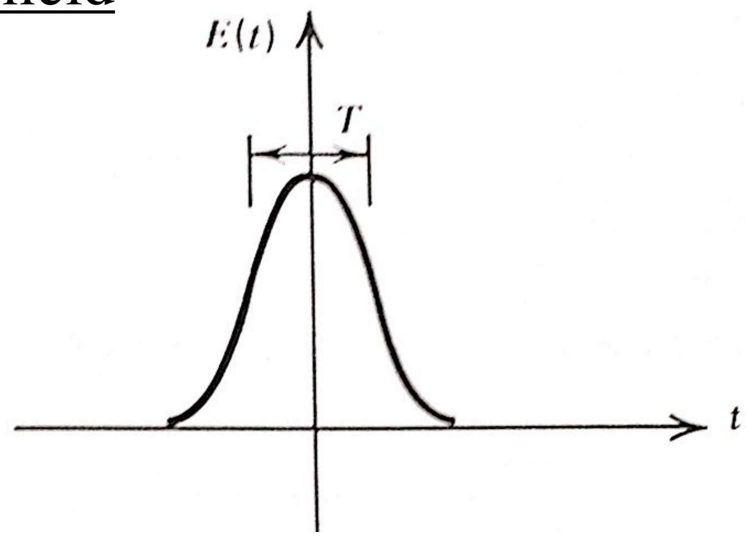
Here, we used Parseval's theorem: $\displaystyle\int_{-\infty}^{\infty} E^2(t)dt = 2\pi\int_{-\infty}^{\infty} |\bar{E}(\omega)|^2 d\omega$

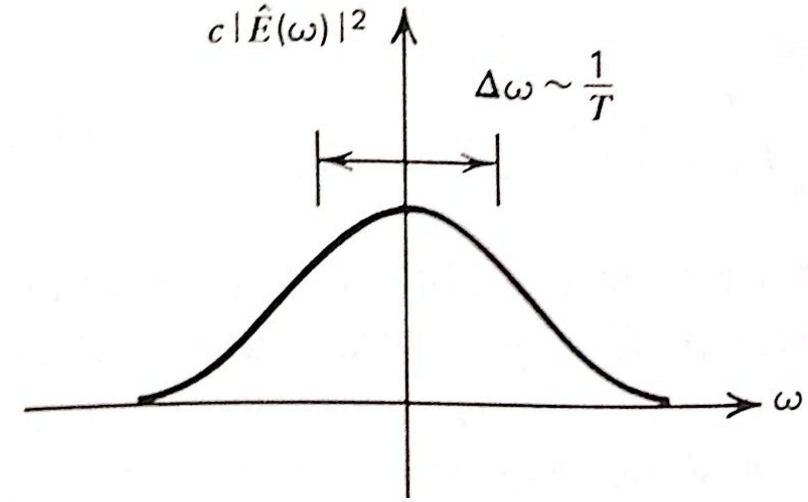
Energy per unit area per unit frequency:

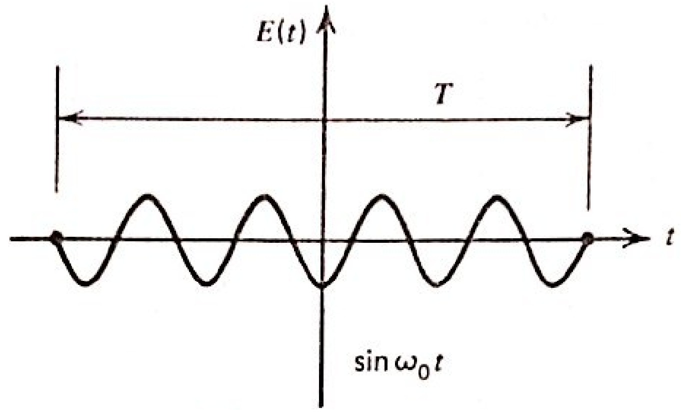
$$\frac{dW}{dAd\omega} = c|\bar{E}(\omega)|^2$$

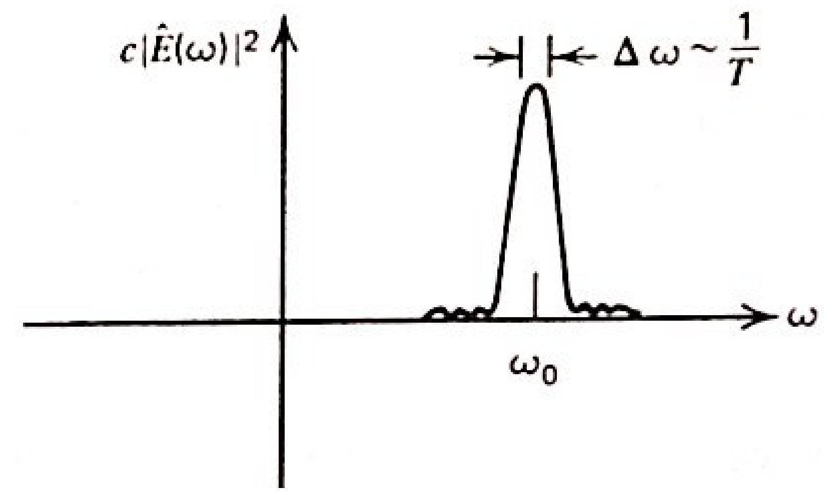
electric field

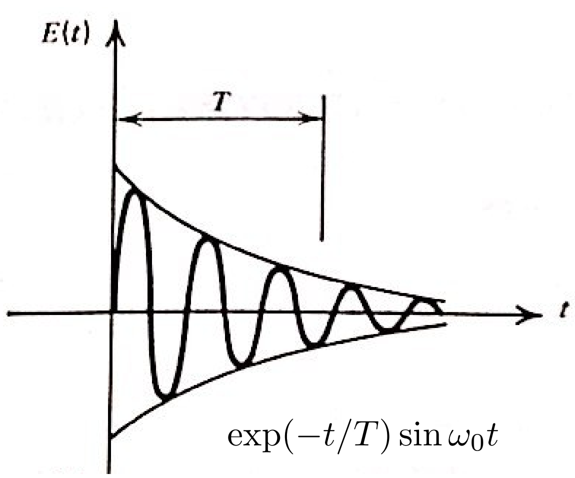
power spectrum

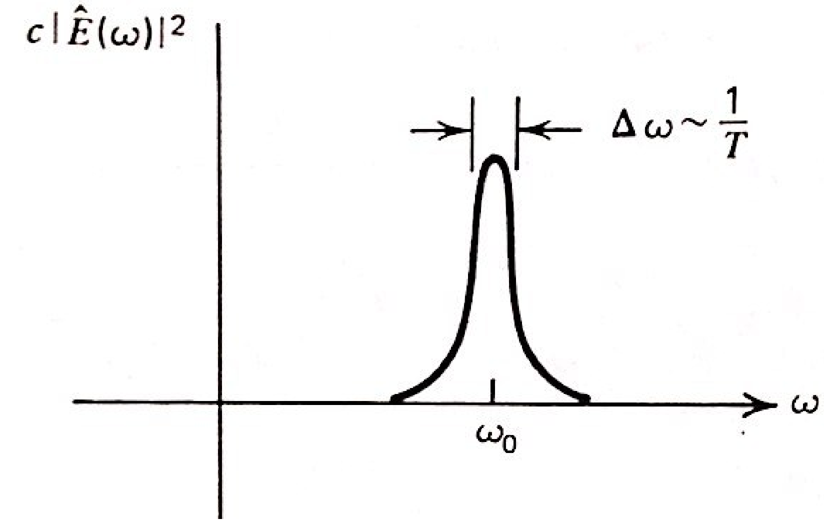












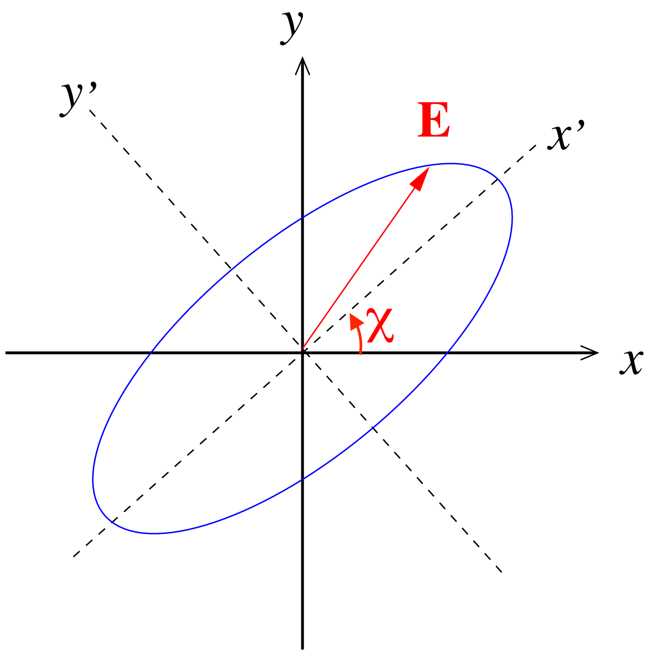
Polarization

- Let us consider a plane EM wave propagating in the +z direction, and examine the electric vector at an arbitrary point (say, $\mathbf{r} = 0$). Because the electric field is transverse, the electric field can be expressed as

$$\mathbf{E} = \hat{\mathbf{x}} E_1 e^{-i\omega t} + \hat{\mathbf{y}} E_2 e^{-i\omega t}$$

Complex amplitudes can be expressed as

$$E_1 = \mathcal{E}_1 e^{i\phi_1} \qquad E_2 = \mathcal{E}_2 e^{i\phi_2} \quad \text{where } \mathcal{E}_1, \mathcal{E}_2, \phi_1, \phi_2 \text{ are real.}$$



Then, the real part of **E** is

$$\mathbf{E} = \hat{\mathbf{x}} \mathcal{E}_1 \cos(\omega t - \phi_1) + \hat{\mathbf{y}} \mathcal{E}_2 \cos(\omega t - \phi_2)$$

As a function of time, the tip of **E** will trace out an ellipse, meaning that the general wave is **elliptically polarized**.

- In general, the principal axes of this ellipse will have a tilt angle χ w.r.t. to x-y axes. We define the zero of time so that **E** lies along the x' direction at $t = 0$.

$$\mathbf{E} = \hat{\mathbf{x}}' \mathcal{E}_1' \cos \omega t + \hat{\mathbf{y}}' \mathcal{E}_2' \sin \omega t$$

- Taking time average of the $|\mathbf{E}|^2$, we obtain:

$$\langle |\mathbf{E}|^2 \rangle = \mathcal{E}_1^2 + \mathcal{E}_2^2 = \mathcal{E}_1'^2 + \mathcal{E}_2'^2 = \text{constant} \equiv \mathcal{E}_0^2$$

- We can satisfy the late part of the above equation by defining an **ellipticity angle**:

$$\mathcal{E}_1' = \mathcal{E}_0 \cos\beta \quad \mathcal{E}_2' = -\mathcal{E}_0 \sin\beta \ \text{ where } \ -\pi/2 \le \beta \le \pi/2 \ \ (\text{or } \ \mathcal{E}_2' = \mathcal{E}_0 \sin\beta', \ \beta' = -\beta)$$

$0 < \beta < \pi/2$	: clockwise (right-handed polarization, negative helicity)
$-\pi/2 < \beta < 0$	: counterclockwise (left-handed polarization, positive helicity)
$\beta = \pm\pi/4$	: circularly polarized
$\beta = 0$ or $\pm\pi/2$	: linearly polarized

- With the relations

$$\begin{aligned}&\hat{\mathbf{x}}\mathcal{E}_1 \cos(\omega t - \phi_1) + \hat{\mathbf{y}}\mathcal{E}_2 \cos(\omega t - \phi_2) \\ &= \hat{\mathbf{x}}'\mathcal{E}_0 \cos\beta \cos\omega t - \hat{\mathbf{y}}'\mathcal{E}_0 \sin\beta \sin\omega t\end{aligned}$$

$+$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

$$\begin{pmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \end{pmatrix} = \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \end{pmatrix}$$

we obtain the relations:

$$\begin{aligned}\mathcal{E}_1 \cos\phi_1 &= \mathcal{E}_0 \cos\beta \cos\chi \\ \mathcal{E}_1 \sin\phi_1 &= \mathcal{E}_0 \sin\beta \sin\chi \\ \mathcal{E}_2 \cos\phi_2 &= \mathcal{E}_0 \cos\beta \sin\chi \\ \mathcal{E}_2 \sin\phi_2 &= -\mathcal{E}_0 \sin\beta \cos\chi\end{aligned}$$

Given $\mathcal{E}_1, \ \phi_1, \ \mathcal{E}_2, \ \phi_2$, we can solve for $\mathcal{E}_0, \ \beta, \ \chi$.

Stokes Parameters (for monochromatic waves)

- A convenient way to solve these equations is by means of the **Stokes parameters for monochromatic waves**.

$$
\begin{aligned}
I &\equiv \mathcal{E}_1^2 + \mathcal{E}_2^2 = \mathcal{E}_0^2 \\
Q &\equiv \mathcal{E}_1^2 - \mathcal{E}_2^2 = \mathcal{E}_0^2 \cos 2\beta \cos 2\chi \\
U &\equiv 2\mathcal{E}_1\mathcal{E}_2 \cos(\phi_1 - \phi_2) = \mathcal{E}_0^2 \cos 2\beta \sin 2\chi \\
V &\equiv 2\mathcal{E}_1\mathcal{E}_2 \sin(\phi_1 - \phi_2) = \mathcal{E}_0^2 \sin 2\beta
\end{aligned}
\longrightarrow \quad I^2 = Q^2 + U^2 + V^2
$$

for a monochromatic wave (pure polarization)

Then, we have

$$
\mathcal{E}_0 = \sqrt{I}, \quad \sin 2\beta = \frac{V}{I}, \quad \tan 2\chi = \frac{U}{Q}
$$

Pure elliptical polarization is determined sole by three parameters $(\mathcal{E}_0,\ \beta,\ \chi)$.

- Meaning of the Stokes parameters:

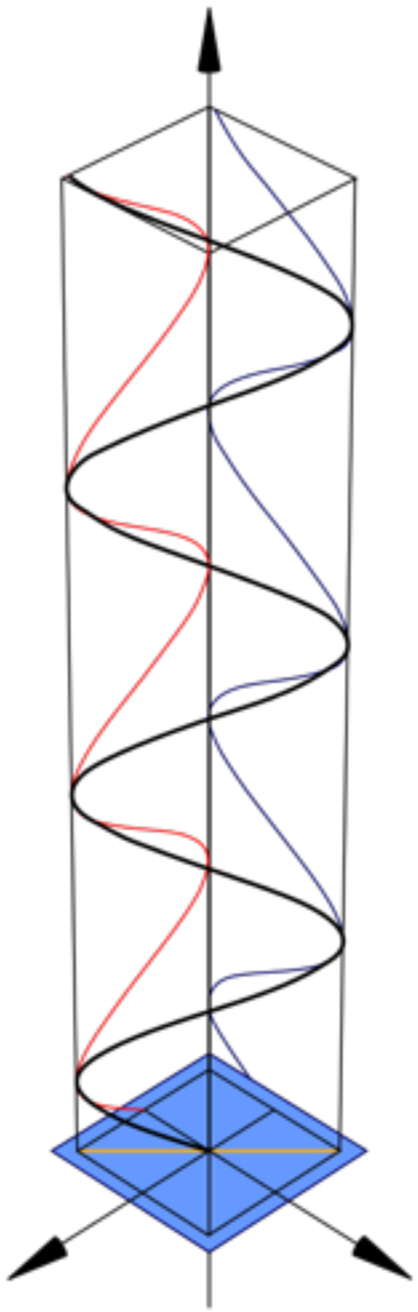
I : total energy flux or intensity

V : circularity parameter ($V > 0$: right-handed, $V < 0$: left-handed)

Q, U : orientation of the ellipse (or line) relative to the x-axis

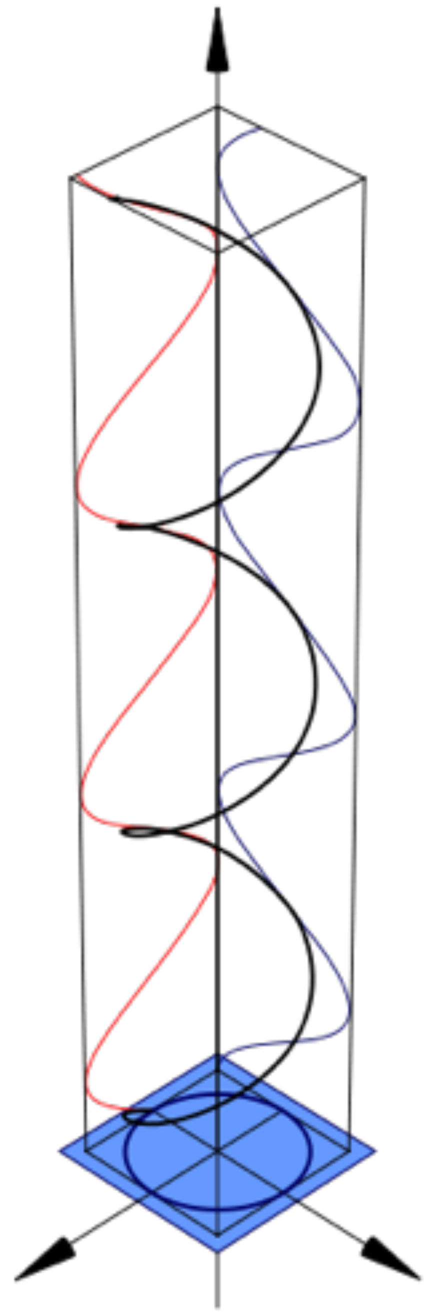
$Q \times U \neq 0, V = 0$	: linear polarization
$Q = U = 0, V \neq 0$	: circular polarization
$Q \times U \neq 0, V \neq 0$	: elliptical polarization

Figures from Wikipedia



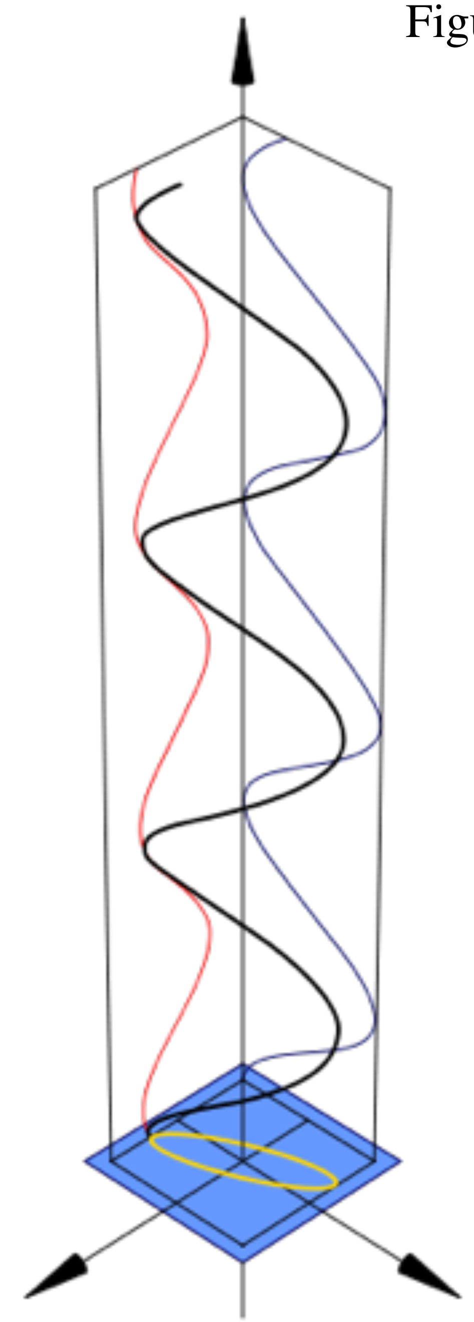
Linear

$$|\phi_1 - \phi_2| = 0$$
$$|\beta| = 0, \pi/2$$
$$\mathcal{E}_1/\mathcal{E}_2 = \text{const.}$$



Circular

$$|\phi_1 - \phi_2| = \pi/2$$
$$|\beta| = \pi/4$$
$$|\mathcal{E}_1/\mathcal{E}_2| = 1$$



Elliptical

$$|\phi_1 - \phi_2| \neq 0, \pi/2$$
$$|\beta| \neq 0, \pi/4, \pi/4$$
$$\mathcal{E}_1/\mathcal{E}_2 \neq \pm 1$$

Stokes Parameters (for quasi-monochromatic waves)

- In general, EM waves vary over time and with wavenumber. Clearly, then, the practical measurement of EM waves involves taking a time average over a time interval.
- Consider EM wave with **slowly varying** amplitudes and phases:

$$E_1(t) = \mathcal{E}_1(t)e^{i\phi_1(t)} \qquad E_2(t) = \mathcal{E}_2(t)e^{i\phi_2(t)}$$

- How slow is slow? **Quasi-monochromatic wave**:

  Assumption: over a time interval $\Delta t > \Delta t_c \equiv 1/\omega$, the amplitudes and phases do not change significantly. By the uncertainty relation, its frequency spread $\Delta\omega$ about the value ω can be estimated as $\Delta\omega/\omega \approx \Delta t_c/\Delta t < 1$.

  For this reason, the wave slowly varying over a time interval $\Delta t > \Delta t_c = 1/\omega$ is called **quasi-monochromatic**, and the time Δt_c is called the **coherence time**.

- The **Stokes parameters for quasi-monochromatic waves** are defined by the following average over time, to be consistent with the definition for monochromatic waves:

$$\boxed{\begin{aligned}
I &\equiv \langle E_1 E_1^* \rangle + \langle E_2 E_2^* \rangle = \left\langle \mathcal{E}_1^2 + \mathcal{E}_2^2 \right\rangle \\
Q &\equiv \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle = \left\langle \mathcal{E}_1^2 - \mathcal{E}_2^2 \right\rangle \\
U &\equiv \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle = \langle 2\mathcal{E}_1 \mathcal{E}_2 \cos(\phi_1 - \phi_2) \rangle \\
V &\equiv \frac{1}{i} \left(\langle E_1 E_2^* \rangle - \langle E_2 E_1^* \rangle \right) = \langle 2\mathcal{E}_1 \mathcal{E}_2 \sin(\phi_1 - \phi_2) \rangle
\end{aligned}}$$

- With the Schwartz inequality $\langle E_1 E_1^* \rangle \langle E_2 E_2^* \rangle \geq \langle E_1 E_2^* \rangle \langle E_2 E_1^* \rangle$

  we can easily verify that

$$\boxed{I^2 \geq Q^2 + U^2 + V^2}$$

  The equality holds only for a completely polarized wave.

- Most sources of EM radiation a large number of atoms or molecules that emit light. The orientation of the electric fields produced by these emitters may not be correlated, in which case the light is said to be **unpolarized**. For completely **unpolarized** wave, where the phase difference $\phi_1 - \phi_2$ between E_1 and E_2 maintain no permanent relation and where there is no preferred orientation in the *x*-*y* plane, so that $\langle \mathcal{E}_1^2 \rangle = \langle \mathcal{E}_2^2 \rangle$.

$$Q = U = V = 0$$

Proof of the inequality: Homework: (1) Derive the Schwartz inequality. (2) Show that $I^2 \geq Q^2 + U^2 + V^2$

Superposition of independent waves

- Radiation will generally originate from a variety of regions different polarizations and different wave phases. Consider therefore a beam consisting of a mixture of many independent waves:

$$E_1 = \sum_k E_1^{(k)} \qquad E_2 = \sum_k E_2^{(k)} \qquad \text{where } k = 1, 2, 3, \cdots .$$

$$\left\langle E_i E_j^* \right\rangle = \sum_k \sum_l \left\langle E_i^{(k)} E_j^{(l)*} \right\rangle = \sum_k \left\langle E_i^{(k)} E_j^{(k)*} \right\rangle \qquad (i, j = 1 \text{ or } 2)$$

Because the relative phases are random, only term $k = l$ survive the averaging. Therefore, the **Stokes parameters have additive properties**:

$$\boxed{I = \sum_k I^{(k)}, \quad Q = \sum_k Q^{(k)}, \quad U = \sum_k U^{(k)}, \quad V = \sum_k V^{(k)}}$$

- By the superposition principle, an arbitrary wave can be decomposed of a completely unpolarized wave and a completely polarized wave.

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} I - \sqrt{Q^2 + U^2 + V^2} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \sqrt{Q^2 + U^2 + V^2} \\ Q \\ U \\ V \end{pmatrix}$$

- Proof of the inequality: $I^2 = (I_{\rm pol} + I_{\rm unpol})^2 \geq I_{\rm pol}^2 = Q^2 + U^2 + V^2$

- **Degree of polarization** for a partially polarized wave = ratio of the intensity of the polarized part to the total intensity

$$\boxed{\Pi \equiv \frac{I_{\rm pol}}{I} = \frac{\sqrt{Q^2+U^2+V^2}}{I}}$$

- In the case of partial **linear polarization (V = 0)**, the measurement consists of rotating a linear polarization filter until the maximum values of intensity are found. The maximum value will occur when the filter is aligned with the plane of polarization, and the minimum value will occur along in the direction perpendicular to it.

  Total value of the unpolarized intensity is shared equally between any two perpendicular directions. Therefore,

$$I_{\rm max} = \frac{1}{2}I_{\rm unpol} + I_{\rm pol} \qquad \text{where} \quad I_{\rm unpol} = I - \sqrt{Q^2+U^2}$$

$$I_{\rm min} = \frac{1}{2}I_{\rm unpol} \qquad I_{\rm pol} = \sqrt{Q^2+U^2}$$

$$\boxed{\therefore \quad \Pi_{\rm linear} = \frac{I_{\rm max}-I_{\rm min}}{I_{\rm max}+I_{\rm min}}}$$

  This equation will underestimate the true degree of polarization if circular or elliptical polarization is present.

$$\begin{aligned} I_{\rm max} &= \tfrac{1}{2}(I_{\rm unpol}+I_{\rm cir}) + I_{\rm lin} \\ I_{\rm min} &= \tfrac{1}{2}(I_{\rm unpol}+I_{\rm cir}) \end{aligned} \quad \rightarrow \quad \frac{I_{\rm max}-I_{\rm min}}{I_{\rm max}+I_{\rm min}} = \frac{I_{\rm lin}}{I} < \frac{I_{\rm pol}}{I} = \frac{I_{\rm lin}+I_{\rm cir}}{I_{\rm unpol}+I_{\rm lin}+I_{\rm cir}}$$

Electromagnetic Potentials

- **Vector potential**: from the vector identity $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, the equation $\nabla \cdot \mathbf{B} = 0$ yields

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}} \qquad \mathbf{A}: \text{ vector potential}$$

Then, $\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \quad \rightarrow \quad \nabla \times \left(\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) = 0$

- **Scalar potential**: from the vector identity $\nabla \times (\nabla\phi) = 0$, this equation can be satisfied if we define a potential such as

$$\boxed{\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}} \qquad \phi: \text{ scalar potential}$$

- **Gauge invariance**:

B will be unchanged for any transformation

$$\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla\psi \qquad \text{since } \nabla \times (\nabla\psi) = 0$$

E will also be unchanged if at the same time the scalar potential is changed by

$$\phi \rightarrow \phi' = \phi - \frac{1}{c}\frac{\partial \psi}{\partial t}$$

EM field is invariant under the **Gauge transform** $\boxed{(\phi,\ \mathbf{A}) \rightarrow \left(\phi - \frac{1}{c}\frac{\partial \psi}{\partial t},\ \mathbf{A} + \nabla\psi\right)}$

Lorentz Gauge

- The Lorentz gauge is the most important gauge in the EM theory, defined by:

$$\boxed{\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0}$$

Note that we can alway choose a function ψ such as:

$$\nabla \cdot \mathbf{A}' + \frac{1}{c}\frac{\partial \phi'}{\partial t} = \nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} + \left(\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}\right) = 0$$

- Maxwell's equations:

$$\nabla \cdot \mathbf{E} = 4\pi\rho \;\rightarrow\; \nabla^2 \phi + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -4\pi\rho$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j} \;\rightarrow\; \nabla \times (\nabla \times \mathbf{A}) - \frac{1}{c}\frac{\partial}{\partial t}\left(-\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) = \frac{4\pi}{c}\mathbf{j}$$

$$\rightarrow\; -\nabla^2 \mathbf{A} + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t}\right) = \frac{4\pi}{c}\mathbf{j}$$

$$\text{Note}: \nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A})$$

- With the Lorentz gauge, the above equations become:

$$\boxed{\begin{aligned} \nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} &= -4\pi\rho \\ \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\frac{4\pi}{c}\mathbf{j} \end{aligned}}$$

Retarded potentials

- The solutions to the above equations are (called the **retarded potentials**, see Jackson):

$$\phi(\mathbf{r}, t) = \int d^3\mathbf{r}' \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|}, \qquad \mathbf{A}(\mathbf{r}, t) = \frac{1}{c} \int d^3\mathbf{r}' \frac{\mathbf{j}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|}$$

$$t' \equiv t_{\rm ret} = t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}$$

- The **retarded time** refers to conditions at the point $\mathbf{r}'$ that existed at a time earlier than t by just the time required for light to travel from $\mathbf{r}'$ to $\mathbf{r}$.
- The potentials responds to the changes after "retarded time" delay.

Applicability of the Radiative Transfer Theory

- We defined specific intensity by the relation: $dE = I_\nu dA d\Omega d\nu dt$

  We should note that dA and $d\Omega$ cannot both be made arbitrarily small because of the uncertainty principle:

$$dx dp_x dy dp_y = p^2 dA d\Omega \geq h^2 \quad \rightarrow \quad dA d\Omega \geq h^2/p^2 = \lambda^2$$

  There is another limitation because of the energy uncertainty principle:

$$dE dt \geq h \quad \rightarrow \quad d\nu dt \geq 1$$

- Therefore, when the wavelength of light is larger than atomic dimensions (Bohr radius, $a_0 = 0.53\text{Å}$), as in the optical, we cannot describe the interaction of light on the atomic scale in terms of specific intensity.
- However, we may still regard transfer theory as a valid macroscopic theory, <u>provided the absorption and emission properties are correctly calculated from microscopic theories</u> (electromagnetic or quantum theory).
- A more precise, classical treatment of the validity of rays is known as the eikonal approximation. (from German “eikonal”, which is from Greek word meaning “image”)