# Radiative Processes in Astrophysics

Lecture 2
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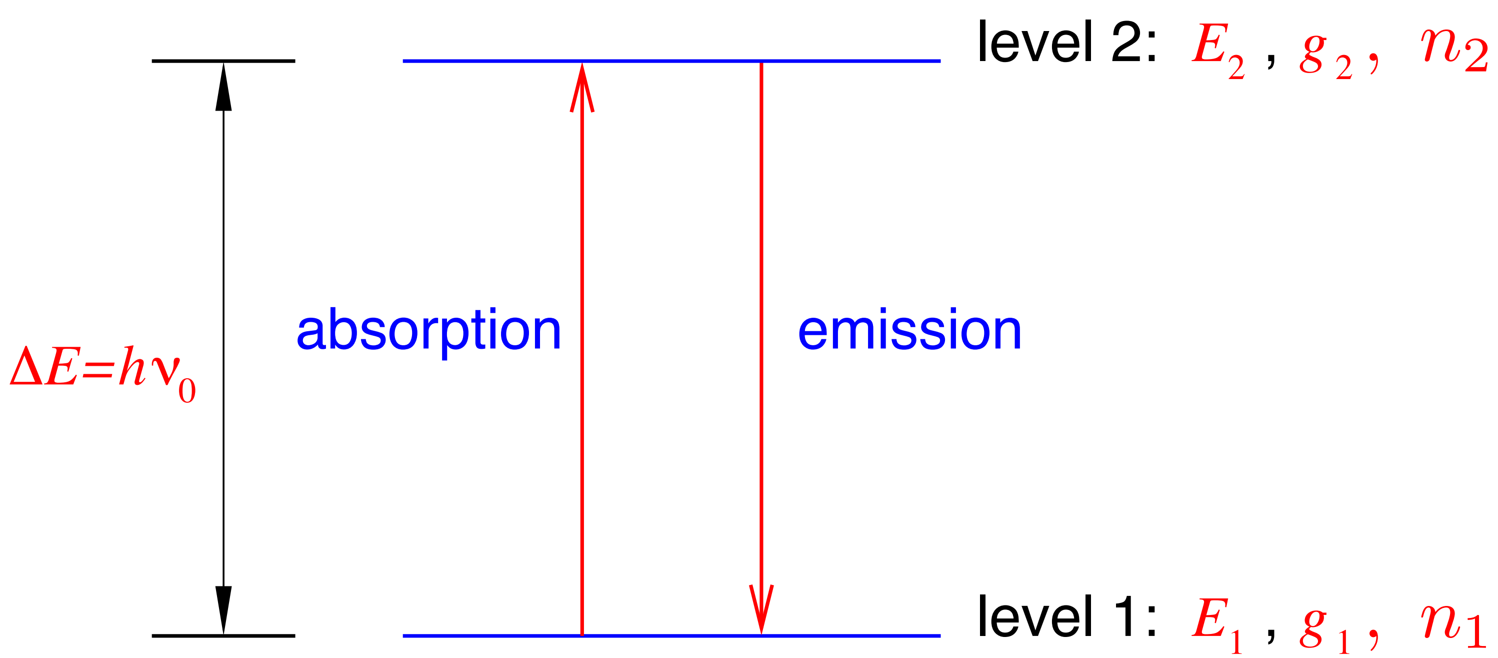
# The Einstein Coefficients

- Consider a system with two discrete energy levels $(E_1, E_2)$ and degeneracies $(g_1, g_2)$. Let $(n_1, n_2)$ be the number densities of atoms in levels (1, 2).

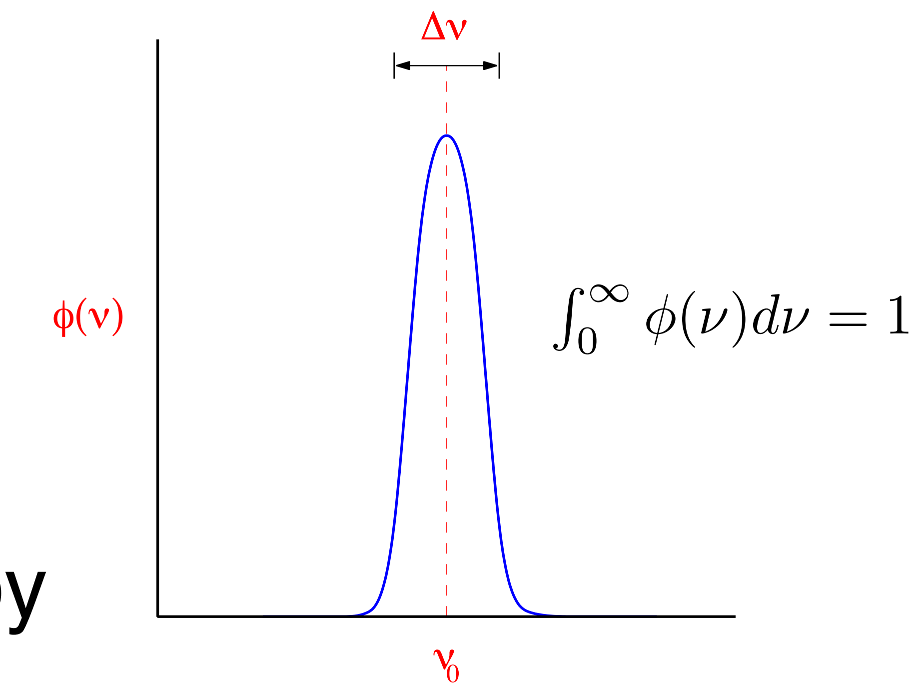



## Spontaneous Emission from level 2 to level 1:

The Einstein A-coefficient $A_{21}$ is the transition probability per unit time for spontaneous emission.

Absorption from level 1 to level 2 occurs in the presence of photons of energy $h\nu_0$ .

- The absorption probability per unit time is proportional to the density of photons (or to the mean intensity) at frequency $\nu_0$ .
- In general, the energy difference between the two levels have finite width which can be described by a line profile function $\phi(\nu)$ .




- The Einstein B-coefficient $B_{12}$ is defined by

$B_{12}\bar{J} =$ transition probability per unit time for absorption

where $\bar{J} = \int_0^\infty J_\nu \phi(\nu) d\nu.$

## Stimulated emission from level 2 to level 1:

- Another Einstein B-coefficient is defined by

  $B_{21}\bar{J} =$ transition probability per unit time for stimulated emission.

- Einstein found that to derive Planck's law another process was required that was proportional to radiation field and caused emission of a photon.
- The stimulated emission is precisely coherent (same direction and frequency, etc) with the photon that induced the emission.

## Note:

- Be aware that the energy density is often used instead of intensity to define the Einstein B-coefficients.

# Relations between Einstein Coefficients

- In TE, total absorption rate = total emission rate:

$$n_1 B_{12} \bar{J} = n_2 A_{21} + n_2 B_{21} \bar{J}$$

$$\rightarrow \bar{J} = \frac{A_{21}/B_{21}}{(n_1/n_2)(B_{12}/B_{21}) - 1}$$

- Populations of the atomic states follow the Boltzmann distribution.

$$\frac{n_1}{n_2} = \frac{g_1 \exp(-E_1/k_{\mathrm{B}}T)}{g_2 \exp(-E_2/k_{\mathrm{B}}T)} = \frac{g_1}{g_2} \exp(h\nu_0/k_{\mathrm{B}}T).$$

- Therefore,

$$\bar{J} = \frac{A_{21}/B_{21}}{(g_1 B_{12}/g_2 B_{21}) \exp(h\nu_0/k_{\mathrm{B}}T) - 1}$$

- In TE, $J_\nu = B_\nu$ for all temperatures. We must have the following Einstein relations:

$$g_1 B_{12} = g_2 B_{21}$$

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21}$$

Einstein relations:

$$g_1 B_{12} = g_2 B_{21}$$
$$A_{21} = \frac{2h\nu^3}{c^2} B_{21}$$

- If we can determine any one of the coefficients, these relations allow us to determine the other two.
- These connect atomic properties $(A_{21}, B_{21}, B_{12})$ and have no reference to the temperature. Thus, the relations must hold whether or not the atoms are in TE.
    - ✦ If the relations were only for TE, the relations would contain the dependence on T.
- Without stimulated emission, Einstein could not get Planck's law, but only Wien's law.
    - ✦ When $h\nu \gg k_{\mathrm{B}}T$ (Wien's limit), level 2 is very sparsely populated relative to level 1. Then, stimulated emission is unimportant compared to absorption.

# Radiative Transfer Equation in terms of Einstein Coefficients

## Emission coefficient:

- assumption: the line profile function of the emitted radiation is the same profile as for the absorption $\phi(\nu)$.
- energy emitted in volume $dV$, solid angle $d\Omega$, frequency range $d\nu$, and time $dt$:

$$j_\nu dV d\Omega d\nu dt = (h\nu/4\pi) n_2 A_{21} dV d\Omega \phi(\nu) d\nu dt$$

  Here, note that each atom emits an energy $h\nu$ distributed over solid angle $4\pi$.

- Then, the emission coefficient is given by

$$\boxed{j_\nu = \frac{h\nu}{4\pi} n_2 A_{21} \phi(\nu)}$$

## Absorption coefficient:

- energy absorbed out of a beam in frequency range $d\nu$, solid angle $d\Omega$, time $dt$, and volume $dV$

$$\alpha_\nu I_\nu dV dt d\Omega d\nu = (h\nu/4\pi) n_1 B_{12} I_\nu dV dt d\Omega \phi(\nu) d\nu$$

- Then, the absorption coefficient (uncorrected for stimulated emission) is given by

$$\alpha_\nu = \frac{h\nu}{4\pi} n_1 B_{12} \phi(\nu)$$

- What about the stimulated emission? It is proportional to the intensity, in close analogy to the absorption process. Thus, the stimulated emission can be treated as negative absorption. The absorption coefficient, corrected for stimulate emission, is

$$\boxed{\alpha_\nu = \frac{h\nu}{4\pi} \phi(\nu) \left( n_1 B_{12} - n_2 B_{21} \right)}$$

## Source function:

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}}$$

- Using the Einstein relations, the absorption coefficient and source function can be written

$$\alpha_\nu = \frac{h\nu}{4\pi} n_1 B_{12} \left(1 - \frac{g_1 n_2}{g_2 n_1}\right)$$

$$\boxed{S_\nu = \frac{2h\nu^3}{c^2} \left(\frac{g_2 n_1}{g_1 n_2} - 1\right)^{-1} \quad \rightarrow \text{ generalized Kirchhoff's law}}$$

# Thermal Emission (LTE)

- If the matter is in TE with itself (but not necessarily with the radiation), we have the Boltzmann distribution. The matter is said to be in LTE.

$$\frac{n_1}{n_2} = \frac{g_1}{g_2} \exp\left(h\nu / k_{\rm B} T\right)$$

- In LTE, we obtain the absorption coefficient and the Kirchhoff's law:

$$\alpha_\nu = \frac{h\nu}{4\pi} n_1 B_{12} \left[1 - \exp\left(-\frac{h\nu}{k_{\rm B} T}\right)\right] \phi(\nu)$$

$$\boxed{S_\nu = B_\nu(T) \ \rightarrow \ \text{Kirchhoff's law in LTE}}$$

- The Kirchhoff's law holds even in LTE condition.

# Normal & Inverted Populations

## Normal populations:

- In LTE, $$\frac{n_2 g_1}{n_1 g_2} = \exp\left(-\frac{h\nu}{k_{\rm B}T}\right) < 1 \;\rightarrow\; \frac{n_1}{g_1} > \frac{n_2}{g_2}$$
- The normal populations is usually satisfied even when the material is out of thermal equilibrium.

## Inverted populations: $\frac{n_1}{g_1} < \frac{n_2}{g_2}$

- In this case, the absorption coefficient is negative and the intensity increases along a ray.
- Such a system is said to be a **maser** (**m**icrowave **a**mplification by **s**timulated **e**mission of **r**adiation; also **laser** for **l**ight...).
- The amplification can be very large. A negative optical depth of -100 leads to an amplification by a factor of $e^{100} = 10^{43}$.

# Scattering Effects: Pure Scattering

- Assumptions

isotropic scattering: scattered equally into equal solid angles

coherent scattering (elastic or monochromatic scattering): the total amount of radiation scattered per unit frequency is equal to the total amount absorbed in the same frequency range.

Thompson scattering (scattering from non-relativistic electrons) is nearly coherent.

- scattering coefficient

In the textbook,
the scattering coefficient is denoted by $\sigma_\nu$ .

$$j_\nu = \alpha_\nu^{\rm sca} \int \Phi_\nu(\boldsymbol{\Omega}, \boldsymbol{\Omega}') I_\nu(\boldsymbol{\Omega}') d\Omega'$$

$$= \alpha_\nu^{\rm sca} \frac{1}{4\pi} \int I_\nu d\Omega = \alpha_\nu^{\rm sca} J_\nu$$

- source function

$$S_\nu = J_\nu = \frac{1}{4\pi} \int I_\nu d\Omega$$

- radiative transfer equation

$$\frac{dI_\nu}{ds} = -\alpha_\nu^{\rm sca}(I_\nu - J_\nu)$$

This is an integro-differential equation, and cannot be solved by the formal solution.

$\rightarrow$ Rosseland approximation, Eddington approximation, or random walks

# Random Walks (in infinite medium)

- Random walks: let's consider a single photon rather than a beam of photons (i.e., ray).
- <u>In an infinite, homogeneous medium</u>, net displacement of the photon after *N* free paths is zero, because the average displacement, being a vector, must be zero.

$$\mathbf{R} = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \cdots + \mathbf{r}_N \quad \rightarrow \quad \langle \mathbf{R} \rangle = 0$$

- root mean square net displacement:

$$\begin{aligned} l_*^2 \equiv \left\langle \mathbf{R}^2 \right\rangle &= \left\langle \mathbf{r}_1^2 \right\rangle + \left\langle \mathbf{r}_2^2 \right\rangle + \left\langle \mathbf{r}_3^2 \right\rangle + \cdots + \left\langle \mathbf{r}_N^2 \right\rangle \\ &\quad + 2\left\langle \mathbf{r}_1 \cdot \mathbf{r}_2 \right\rangle + 2\left\langle \mathbf{r}_1 \cdot \mathbf{r}_3 \right\rangle + \cdots \\ &\approx N l^2 \leftarrow \quad \text{Note} \quad \left\langle \mathbf{r}_i^2 \right\rangle \approx l^2, \quad \left\langle \mathbf{r}_i \cdot \mathbf{r}_j \right\rangle = 0 \;\; (i \neq j) \end{aligned}$$

$$\therefore \; l_* = \sqrt{N} l$$

The cross terms involve averaging the cosine of the angle between the directions before and after scattering, and this vanishes for isotropic scattering and for any scattering with front-back symmetry (Thompson or Rayleigh scattering)

# Random Walks (in finite medium)

- In a finite medium, a photon generated somewhere within the medium will scatter until it escapes completely.
- For regions of large optical depth, the mean number of scatterings to escape is roughly determined by $l_* \approx L$ (the typical size of the medium).

$$l_* = \sqrt{N} l \approx L \;\rightarrow\; N \approx L^2/l^2 = L^2 (n\sigma_\nu^{\rm sca})^2$$

$$\therefore\; N \approx \tau^2 \quad (\tau \gg 1)$$

- For regions of small optical depth, the probability of scatterings within $\tau$ is $1 - e^{-\tau} \approx \tau$.

$$\therefore\; N \approx \tau \quad (\tau \ll 1)$$

- For any optical thickness, the mean number of scatterings is

$$N \approx \tau^2 + \tau \quad \text{or} \quad N \approx \max(\tau,\ \tau^2)$$

# Combined Scattering and Absorption

- The transfer equation to the case of combined absorption and scattering.

$$\frac{dI_\nu}{ds} = -\alpha_\nu^{\rm abs}(I_\nu - B_\nu) - \alpha_\nu^{\rm sca}(I_\nu - J_\nu)$$
$$= -(\alpha_\nu^{\rm abs} + \alpha_\nu^{\rm sca})(I_\nu - S_\nu) = -\alpha_\nu^{\rm ext}(I_\nu - S_\nu)$$
$$\text{where} \quad S_\nu \equiv \frac{\alpha_\nu^{\rm abs} B_\nu + \alpha_\nu^{\rm sca} J_\nu}{\alpha_\nu^{\rm abs} + \alpha_\nu^{\rm sca}} \quad \text{and} \quad \alpha_\nu^{\rm ext} \equiv \alpha_\nu^{\rm abs} + \alpha_\nu^{\rm sca}$$

- Source function is an weighted average of the two source functions.
- extinction coefficient: $\alpha_\nu^{\rm ext} \equiv \alpha_\nu^{\rm abs} + \alpha_\nu^{\rm sca}$
- optical depth: $d\tau_\nu \equiv \alpha_\nu^{\rm ext} ds$
- If a matter element is deep inside a medium (i.e., in TE),

$$J_\nu = B_\nu \;\rightarrow\; S_\nu = B_\nu$$

- If the element is isolate in free space, $J_\nu = 0 \;\rightarrow\; S_\nu = \alpha_\nu^{\rm abs} B_\nu / \alpha_\nu^{\rm ext}$

- generalized mean free path:

$$l_\nu = (\alpha_\nu^{\rm abs} + \alpha_\nu^{\rm sca})^{-1}$$

- probability of a (random walk) step ending in absorption:

$$\epsilon_\nu = \alpha_\nu^{\rm abs}/\alpha_\nu^{\rm ext}$$

- probability for scattering (known as the single-scattering albedo)

$$a_\nu = 1 - \epsilon_\nu = \alpha_\nu^{\rm sca}/\alpha_\nu^{\rm ext}$$

- source function:

$$\begin{aligned} S_\nu &= \epsilon_\nu B_\nu + (1-\epsilon_\nu) J_\nu \\ &= (1-a_\nu) B_\nu + a_\nu J_\nu \end{aligned}$$

# Random Walks with Scattering and Absorption

- In an infinite medium, every photon is eventually absorbed.
- Since a random walk can be terminated with probability $\epsilon \; (= \alpha^{\rm abs}/\alpha^{\rm ext})$ at the end of each free path, the mean number of free paths is given by

mean number of free paths x probability of termination = 1

$$N\epsilon = 1 \;\; \rightarrow \;\; N = 1/\epsilon$$

- diffusion length (thermalization length, effective mean path, or effective free path): a measure of the net displacement between the points of creation and destruction of a typical photon.

$$\begin{aligned} l_* &\approx \sqrt{N} l = l/\sqrt{\epsilon} \\ &\approx (\alpha_\nu^{\rm ext})^{-1} \sqrt{\alpha_\nu^{\rm ext}/\alpha_\nu^{\rm abs}} \\ &\approx (\alpha_\nu^{\rm abs} \alpha_\nu^{\rm ext})^{-1/2} \end{aligned}$$

- In a finite medium:
- The behavior depends on wether its size L is larger or smaller than the effective free path $l_*$ .
- effective optical thickness: $\tau_* = L/l_* \approx \sqrt{\tau_{\rm abs}(\tau_{\rm abs} + \tau_{\rm sca})} = \sqrt{\tau_{\rm abs}\tau_{\rm ext}}$

$$\text{where } \tau_{\rm abs} \equiv \alpha_\nu^{\rm abs} L, \ \tau_{\rm sca} \equiv \alpha_\nu^{\rm sca} L, \ \tau_{\rm ext} \equiv \alpha_\nu^{\rm ext}$$

- If effectively thin or translucent $(\tau_* \ll 1, \ L \ll l_*)$, most photons will escape the medium before being destroyed.

luminosity of thermal source with volume V is

$$L_\nu = 4\pi j_\nu V = 4\pi \alpha_\nu B_\nu V \ (\tau_* \ll 1)$$

- If effectively thick, we expect $I_\nu \to B_\nu, \ S_\nu \to B_\nu$, and only the photons emitted within an effective path length of the boundary will have a reasonable chance of escaping before being absorbed.

$$L_\nu = \pi \alpha_\nu^{\rm abs} B_\nu A l_* = \pi \sqrt{\epsilon_\nu} B_\nu A \qquad (F = \pi B \text{ at surface of the source})$$

# Approximate Solutions

How to solve the radiative transfer equation:

$$\frac{dI_\nu}{ds} = -\alpha_\nu^{\rm ext}(I_\nu - S_\nu)$$

$$S_\nu = (1-\epsilon_\nu)J_\nu + \epsilon_\nu B_\nu \text{ and } \epsilon = \alpha_\nu^{\rm abs}/\alpha_\nu^{\rm ext}$$

We will learn two approximations to solve the equation.

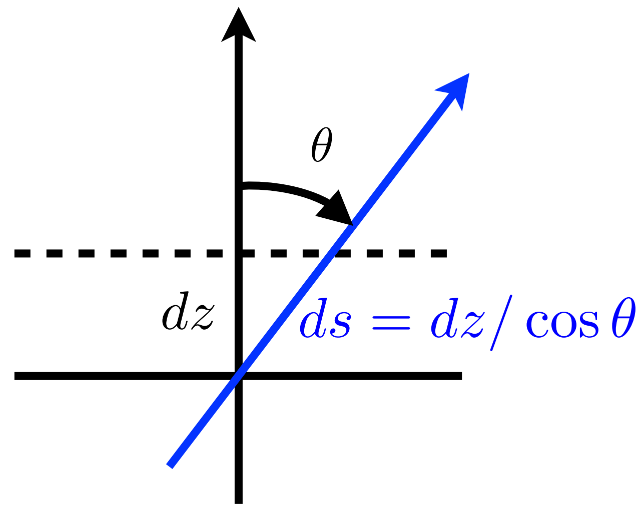
- Rosseland approximation
- Eddington approximation

# Radiative Diffusion: (1) Rosseland Approximation

- Imagine a plane-parallel medium (in which $\rho, T$ depend only on depth *z*).

$$ds = \frac{dz}{\cos\theta} = \frac{dz}{\mu} \rightarrow \mu \frac{\partial I_\nu(z,\mu)}{\partial z} = -\alpha_\nu^{\rm ext}(I_\nu - S_\nu)$$

$$I_\nu(z,\mu) = S_\nu - \frac{\mu}{\alpha_\nu^{\rm ext}} \frac{\partial I_\nu}{\partial z}$$




- "zeroth" approximation: when the point in question is deep in the material, all quantities changes slowly on the scale of a mean free path $(l_* = 1/\alpha_\nu^{\rm ext})$ and the derivative term above is very small.

$$I_\nu^{(0)}(z,\mu) \approx S_\nu^{(0)}(T)$$

This is independent of the angle. $\therefore\ J_\nu^{(0)} = S_\nu^{(0)}$ and $I_\nu^{(0)} = S_\nu^{(0)} = B_\nu$

- "first" approximation:

$$I_\nu^{(1)}(z,\mu) \approx S_\nu^{(0)} - \frac{\mu}{\alpha_\nu^{\rm ext}} \frac{\partial I_\nu^{(0)}}{\partial z} = B_\nu(T) - \frac{\mu}{\alpha_\nu^{\rm ext}} \frac{\partial B_\nu(T)}{\partial z} \rightarrow \text{linear in } \mu$$

- net specific flux along *z*: the angle-independent part of the intensity does not contribute to the flux.

$$F_\nu(z) = \int I_\nu^{(1)}(z,\mu)\cos\theta d\Omega = 2\pi \int_{-1}^{+1} I_\nu^{(1)}(z,\mu)\mu d\mu$$

$$= -\frac{2\pi}{\alpha_\nu^{\rm ext}} \frac{\partial B_\nu}{\partial z} \int_{-1}^{+1} \mu^2 d\mu$$

$$= -\frac{4\pi}{3\alpha_\nu^{\rm ext}} \frac{\partial B_\nu(T)}{\partial T} \frac{\partial T}{\partial z}$$

- total integrated flux: $$F(z) = \int_0^\infty F_\nu(z) d\nu = -\frac{4\pi}{3}\frac{\partial T}{\partial z}\int_0^\infty \frac{1}{\alpha_\nu^{\rm ext}}\frac{\partial B_\nu}{\partial T} d\nu$$

let's define the Rosseland mean absorption coefficient

$$\frac{1}{\alpha_R} \equiv \frac{\int_0^\infty \frac{1}{\alpha_\nu^{\rm ext}}\frac{\partial B_\nu}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu}$$

use

$$\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu = \frac{\partial}{\partial T} \int_0^\infty B_\nu d\nu = \frac{\partial \left(\sigma T^4/\pi\right)}{\partial T} = \frac{4\sigma T^3}{\pi}$$

Then, we obtain the Rosseland approximation to radiative flux

$$F(z) = -\frac{16\sigma T^3}{3\alpha_R}\frac{\partial T}{\partial z} \to -\chi \nabla T$$

which is also called the equation of radiative diffusion.

- The flux equation can be interpreted as a heat conduction with an "effective heat conductivity," $\chi = 16\sigma T^3/3\alpha_R$.
- At which frequencies the Rosseland mean becomes important?

The mean involves a weighted average of $1/\alpha_\nu^{\mathrm{ext}}$ so that frequencies at which the extinction coefficient is small (transparent) tend to dominate.

The weighting function $\partial B_\nu/\partial T$ has a shape similar to that of the Planck function, but it peaks at $h\nu_{\mathrm{max}} = 3.8k_{\mathrm{B}}T$, instead of $h\nu_{\mathrm{max}} = 2.8k_{\mathrm{B}}T$.

# Radiative Diffusion: (2) Eddition Approximation

- In Eddington approximation, the intensities are assumed to approach isotropy, and not necessarily their thermal values.

  In the Rosseland approximation, the intensities approach the Planck function at large effective depths.

- Near isotropy can be introduced by assuming that the intensity is linear in $\mu$. (frequency is suppressed for convenience)

$$I(\tau, \mu) = a(\tau) + b(\tau)\mu$$

- Let us take the first three moments.

mean intensity: $$J \equiv \frac{1}{2}\int_{-1}^{+1} I d\mu = a$$

flux: $$H \equiv \frac{1}{2}\int_{-1}^{+1} \mu I d\mu = \frac{b}{3}$$

radiation pressure: $$K \equiv \frac{1}{2}\int_{-1}^{+1} \mu^2 I d\mu = \frac{a}{3}$$

$$\rightarrow \boxed{K = \frac{1}{3}J}$$

Eddington approximation

Compare with the following equations for the isotropic radiation.

$$p = \frac{1}{3}u \quad \left(p \equiv \frac{1}{c}\int I\cos^2\theta d\Omega, \ \ u(\mathbf{\Omega}) = \frac{1}{c}I\right)$$

optical depth and the transfer equation: $d\tau(z) \equiv -\alpha^{\rm ext} dz, \quad \mu \frac{\partial I}{\partial \tau} = I - S$

Note: source function is independent to $\mu$ (because $S = (1-\epsilon)J + \epsilon B$).

Integrate the above equation and obtain the following equations.

$$\frac{1}{2}\int_{-1}^{+1} d\mu \left(\mu \frac{\partial I}{\partial \tau} = I - S\right) \rightarrow \frac{\partial H}{\partial \tau} = J - S$$

$$\frac{1}{2}\int_{-1}^{+1} d\mu\mu \left(\mu \frac{\partial I}{\partial \tau} = I - S\right) \rightarrow \frac{\partial K}{\partial \tau} = H \rightarrow \frac{1}{3}\frac{\partial J}{\partial \tau} = H$$

The two equations can be combined to yield:

$$\frac{1}{3}\frac{\partial^2 J}{\partial \tau^2} = J - S \quad \rightarrow \boxed{\frac{1}{3}\frac{\partial^2 J}{\partial \tau^2} = \epsilon(J - B)}$$

Let us define a new optical depth $\tau_* \equiv \sqrt{3\epsilon}\tau = \sqrt{3\tau_{\rm abs}(\tau_{\rm abs} + \tau_{\rm sca})}$

The radiative equation is then $\boxed{\frac{\partial^2 J}{\partial \tau_*^2} = J - B}$

This equation is sometimes called the radiative diffusion equation. Given the temperature structure of the medium, $B(\tau)$, the equation can be solved for J.

To solve the second order differential equation, we need two boundary conditions. The boundary conditions can be provided in several ways. One way to do is to use two-stream approximation, in which the entire radiation field is represented by radiation at just two angles, i.e., $\mu = \pm\mu_0$ :

$$I(\tau, \mu) = I^+(\tau)\delta(\mu - \mu_0) + I^-(\tau)\delta(\mu + \mu_0)$$

The two terms denote the outward and inward intensities. Then, the three moments are

$$J = \frac{1}{2}\left(I^+ + I^-\right)$$

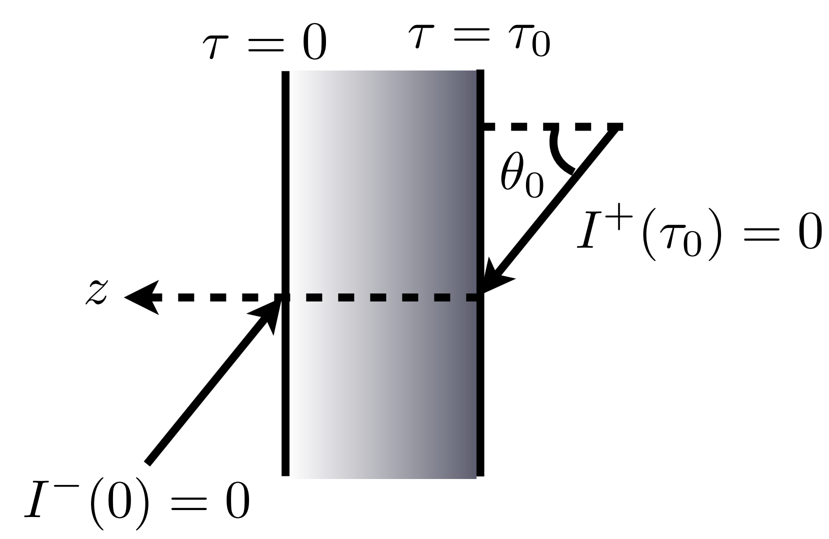
$$H = \frac{1}{2}\mu_0\left(I^+ - I^-\right) \quad \rightarrow \text{ we obtain } \mu_0 = \frac{1}{\sqrt{3}} \text{ in order to satisty } K = \frac{1}{3}J$$

$$K = \frac{1}{2}\mu_0^2\left(I^+ + I^-\right) \qquad (\theta_0 = \cos^{-1}\mu_0 = 54.74^\circ)$$

Using $H = \dfrac{1}{3}\dfrac{\partial J}{\partial \tau}$ , we obtain $\boxed{I^+ = J + \dfrac{1}{3}\dfrac{\partial J}{\partial \tau}, \quad I^- = J - \dfrac{1}{3}\dfrac{\partial J}{\partial \tau}}$.

Suppose the medium extends from $\tau = 0$ to $\tau = \tau_0$ and there is no incident radiation. Then, we obtain two boundary conditions:

$$I^+(\tau_0) = 0 \text{ and } I^-(0) = 0 \quad \rightarrow \quad \boxed{\begin{array}{ll} \dfrac{1}{\sqrt{3}}\dfrac{\partial J}{\partial \tau} = J & \text{at } \tau = 0 \\ \dfrac{1}{\sqrt{3}}\dfrac{\partial J}{\partial \tau} = -J & \text{at } \tau = \tau_0 \end{array}}$$



# Iteration Method

- Recall
$$\frac{dI(s)}{ds} = -\alpha^{\rm ext} I(s) + \alpha^{\rm sca} \int \Phi(\Omega,\Omega') I(s,\Omega') d\Omega' + j(s)$$
or
$$\frac{dI(\tau)}{d\tau} = -I(\tau) + a \int \Phi(\Omega,\Omega') I(\tau,\Omega') d\Omega' + \mathsf{S}(\tau) \qquad \left( d\tau \equiv \alpha^{\rm ext} ds,\ \mathsf{S}(\tau) \equiv \frac{j(\tau)}{\alpha^{\rm ext}} \right)$$

- Let $I_0$ be the intensity of photons that come directly from the source, $I_1$ the intensity of photons that have been scattered once by dust, and $I_n$ the intensity after *n* scatterings. Then,
$$I(s) = \sum_{n=0}^{\infty} I_n(s)$$

- The intensities $I_n$ satisfy the equations.
$$\frac{dI_0(\tau)}{d\tau} = -I_0(\tau) + \mathsf{S}(\tau)$$
$$\frac{dI_n(\tau)}{d\tau} = -I_n(\tau) + a \int \Phi(\Omega,\Omega') I_{n-1}(\tau,\Omega') d\Omega'$$
$$\equiv -I_n(\tau) + \mathsf{S}_{n-1}(\tau) \qquad \left( \mathsf{S}_{n-1}(\tau) \equiv a \int \Phi(\Omega,\Omega') I_{n-1}(\tau,\Omega') d\Omega' \right)$$

- Then, the formal solutions are:
$$\rightarrow \quad I_0(\tau) = e^{-\tau} I_0(0) + \int_0^{\tau} e^{-(\tau-\tau')} \mathsf{S}(\tau') d\tau'$$
$$I_n(\tau) = e^{-\tau} I_n(0) + \int_0^{\tau} e^{-(\tau-\tau')} \mathsf{S}_{n-1}(\tau') d\tau'$$

# Approximation: (1) application to the edge-on galaxies

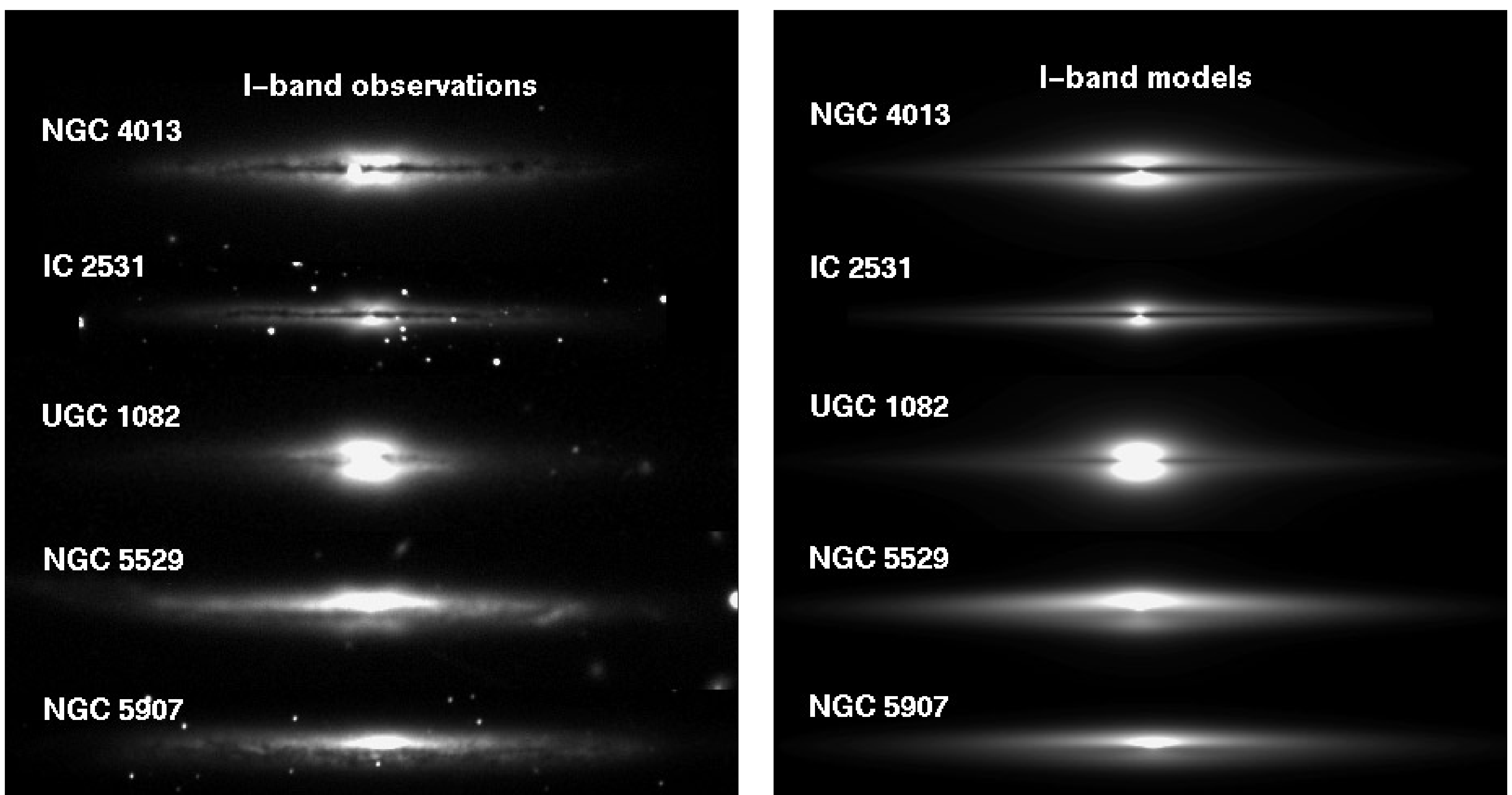
The solution can be further simplified by assuming that

$$\frac{I_n}{I_{n-1}} \approx \frac{I_1}{I_0} \quad (n \geq 2)$$

Then, the infinite series becomes

$$I_n \approx I_0 \sum_{n=0}^{\infty} \left( \frac{I_1}{I_0} \right)^n = \frac{I_0}{1 - I_1/I_0}$$

Kylafis & Bahcall (1987) and Xilouris et al. (1997, 1998, 1999) applied this approximation to model the dust radiative transfer process in the edge-on galaxies.

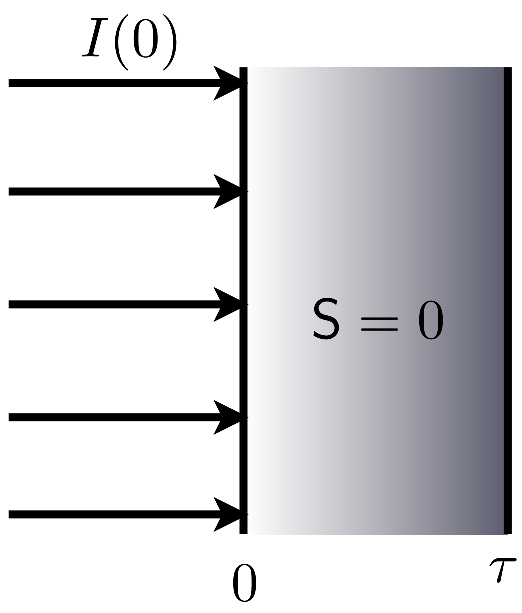



Xilouris et al. (1999)

# Approximation: (2) solution for the forward scattering

- Assume the very strong forward-scattering

$$\Phi(\Omega, \Omega') = \delta(\Omega' - \Omega)$$
$$\rightarrow \quad \mathsf{S}_{n-1}(\tau) = aI_{n-1}(\tau)$$




- The iterative solutions are:

$$I_0(\tau) = e^{-\tau} I_0(0)$$
$$\rightarrow \quad \mathsf{S}_0(\tau) = aI_0(\tau) = ae^{-\tau} I_0(0)$$
$$I_1(\tau) = e^{-\tau} \int_0^\tau e^{\tau'} \mathsf{S}_0(\tau') d\tau' = (a\tau) e^{-\tau} I_0(0)$$
$$\rightarrow \quad \mathsf{S}_1(\tau) = aI_1(\tau) = (a^2\tau) e^{-\tau} I_0(0)$$
$$I_2(\tau) = e^{-\tau} \int_0^\tau e^{\tau'} \mathsf{S}_1(\tau') d\tau' = \frac{(a\tau)^2}{2} I_0(0)$$
$$\rightarrow \quad \mathsf{S}_2(\tau) = aI_2(\tau) = (a^3\tau^2) e^{-\tau} I_0(0)$$
$$I_3(\tau) = e^{-\tau} \int_0^\tau e^{\tau'} \mathsf{S}_2(\tau') d\tau' = \frac{(a\tau)^3}{3 \times 2} I_0(0)$$
$$\vdots$$
$$\rightarrow \quad \mathsf{S}_{n-1}(\tau) = aI_{n-1}(\tau) = (a^n\tau^{n-1}) e^{-\tau} I_0(0)$$
$$I_n(\tau) = e^{-\tau} \int_0^\tau e^{\tau'} \mathsf{S}_{n-1}(\tau') d\tau' = \frac{(a\tau)^n}{n!} I_0(0)$$

The final solutions are:

$$I^{\rm direc}(\tau) = e^{-\tau} I(0)$$
$$\begin{aligned} I^{\rm scatt}(\tau) &= \sum_{n=1}^{\infty} I_n(\tau) = \sum_{n=1}^{\infty} \frac{(a\tau)^n}{n!} e^{-\tau} I(0) \\ &= (e^{a\tau} - 1) e^{-\tau} I(0) \\ &\approx a\tau e^{-\tau} I(0) \quad \text{if} \quad a\tau \ll 1 \\ &\approx a\tau I(0) \quad \text{if} \quad \tau \ll 1 \end{aligned}$$

$$\begin{aligned} I^{\rm tot}(\tau) &= I^{\rm direc}(\tau) + I^{\rm scatt}(\tau) \\ &= e^{-(1-a)\tau} I(0) \\ &= e^{-\tau_{\rm abs}} I(0) \end{aligned}$$