Radiative Processes in Astrophysics

Lecture 1
Sep 4, 2013

Kwang-il Seon
UST / KASI

Overview

Radiative processes

- link astrophysical systems with astronomical observables
- cover many areas of physics and astrophysics (electrodynamics, quantum mechanics, statistical mechanics, relativity...)

Textbooks

- Radiative Processes in Astrophysics (George Rybicki & Alan Lightman)
- The Physics of Interstellar and Intergalactic Medium (Bruce T. Draine)
- The Physics of Astrophysics, Volume 1 Radiation (Frank H. Shu)
- Physics and Chemistry of the Interstellar Medium (Sun Kwok)

1. Fundamentals of Radiative Transfer

Electromagnetic Radiation

Particle/wave duality

classically: electromagnetic waves

- speed of light: $c = 3 \times 10^{10}\mathrm{cm\ s}^{-1}$
- wavelength and frequency: $\lambda = c/\nu$

quantum mechanically: photons

- quanta: massless, spin-1 particles (boson)
- Plank: $E = h\nu = hc/\lambda \ (h = 6.625 \times 10^{-27}\ \mathrm{ergs})$
- Einstein: $E^2 = (m_\gamma c^2)^2 + (pc)^2$

$$p = E/c$$

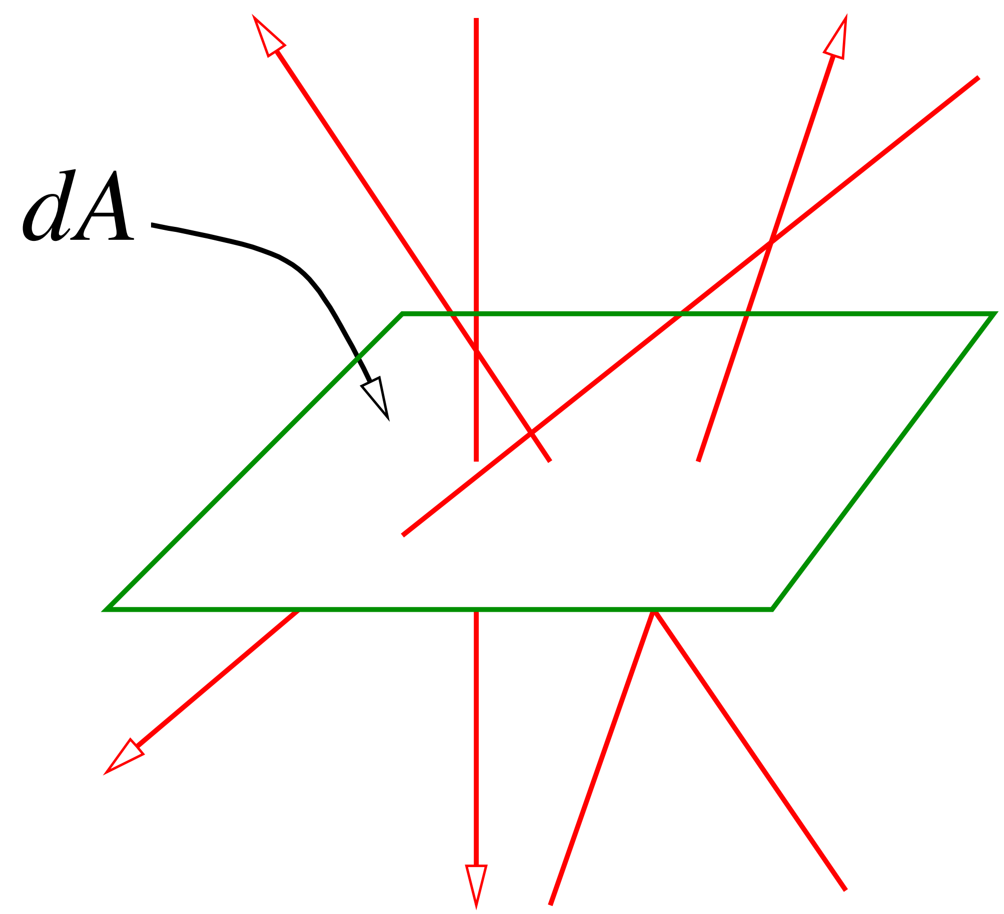
Energy Flux

Definition

- Energy flux, F, is defined as the energy dE passing through an element of area dA in time interval dt

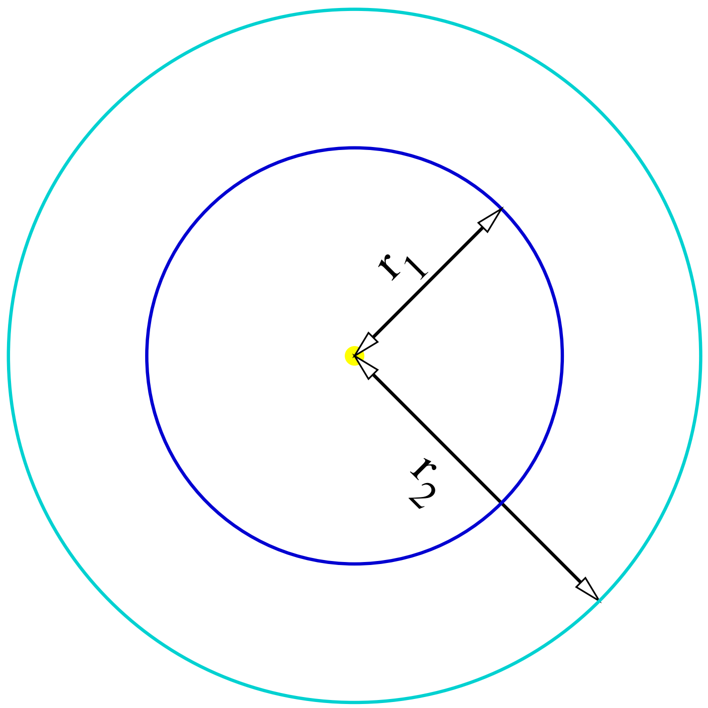
$$dE = F dA dt$$

- F depends on the orientation of elements of dA and the frequency (or wavelength).
- Unit: $\mathrm{erg\ cm^{-2}\ s^{-1}}$



Inverse Square Law

Flux from an isotropic radiation source, i.e., a source emitting equal amounts of energy in all directions.



- Because of energy conservation, flux through two shells around the source must be the same.

$$4\pi r_1^2 F(r_1) = 4\pi r_2^2 F(r_2)$$

- Therefore, we obtain the inverse square law.

$$F = \frac{\text{const.}}{r^2}$$

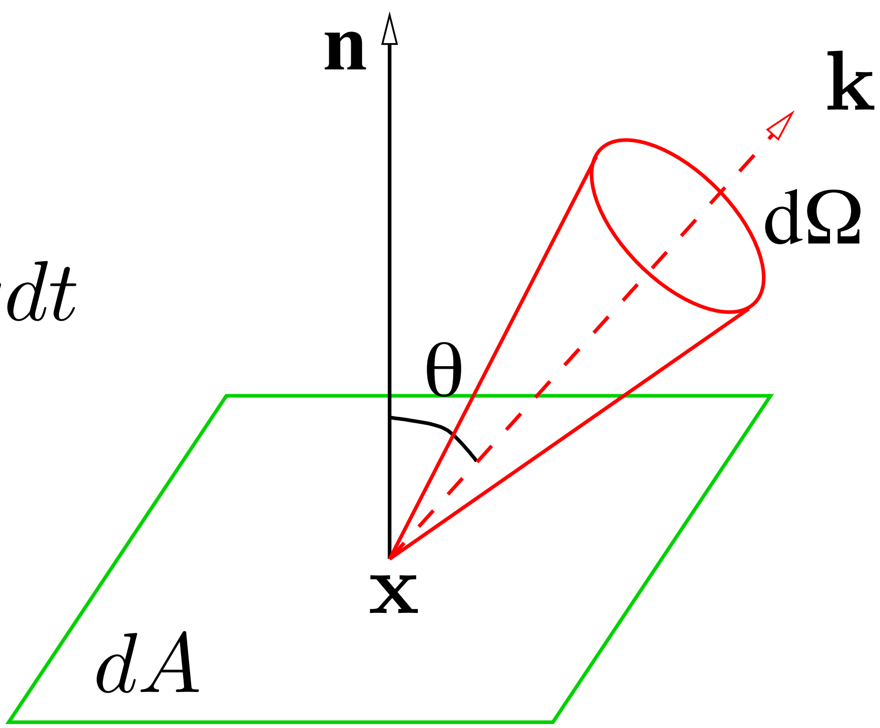
(Specific) Intensity or (Surface) Brightness

Flux = a measure of the energy carried by all rays passing through a given area

Intensity = the energy carried along by individual rays.

- Let dE_ν be the amount of radiant energy which crosses in time dt the area dA with unit normal $\mathbf{n}$ in a direction within solid angle $d\Omega$ centered about $\mathbf{k}$ with photon frequency ν between $\nu + d\nu$.
- The monochromatic specific intensity I_ν is then defined by the equation.

$$\begin{aligned} dE_\nu &= I_\nu(\mathbf{k}, \mathbf{x}, t) dA_\mathbf{k} d\Omega d\nu dt \\ &= I_\nu(\mathbf{k}, \mathbf{x}, t) \cos\theta dA d\Omega d\nu dt \end{aligned}$$



- area normal to $\mathbf{k}$: $dA_\mathbf{k} = dA\cos\theta$
- Unit: $\mathrm{erg\ s^{-1}cm^{-2}sr^{-1}Hz^{-1}}$

Net flux and Momentum flux

Net flux in the direction $\mathbf{n}$ is obtained by integrating the differential flux over all solid angle.

$$F_\nu = \int dF_\nu = \int I_\nu \cos\theta d\Omega$$

- The net flux is zero, if the radiation is isotropic.

$$\begin{aligned} F_\nu &= \int_0^{2\pi}\int_0^{\pi} I_\nu \cos\theta \sin\theta d\theta d\phi = 2\pi I_\nu \int_0^{\pi} \cos\theta \sin\theta d\theta \\ &= \pi I_\nu \left[\sin^2\theta\right]_0^{\pi} = 0 \end{aligned}$$

Momentum flux $\quad p_\nu(\mathrm{dynes\ cm^{-2}\ Hz^{-1}}) = \frac{1}{c}\int I_\nu \cos^2\theta d\Omega$

- The first cosine factor is due to the area normal to $\mathbf{k}$ and the second one is due to the projection of the differential flux vector to the normal vector $\mathbf{n}$.

Note

Moments of intensity

- intensity : scalar (amplitude of the differential flux)
- differential flux : vector
- momentum flux (radiation pressure) : tensor

Intensity can be defined as per wavelength interval.

$$I_\nu |d\nu| = I_\lambda |d\lambda| \qquad \leftarrow \left|\frac{d\nu}{d\lambda}\right| = \frac{c}{\lambda^2} = \frac{\nu}{\lambda}$$
$$\nu I_\nu = \lambda I_\lambda$$

Integrated intensity is defined as the intensity over all frequencies.

$$I = \int_0^\infty I_\nu d\nu = \int_0^\infty I_\lambda d\lambda$$

Constancy of Specific Intensity in Free Space

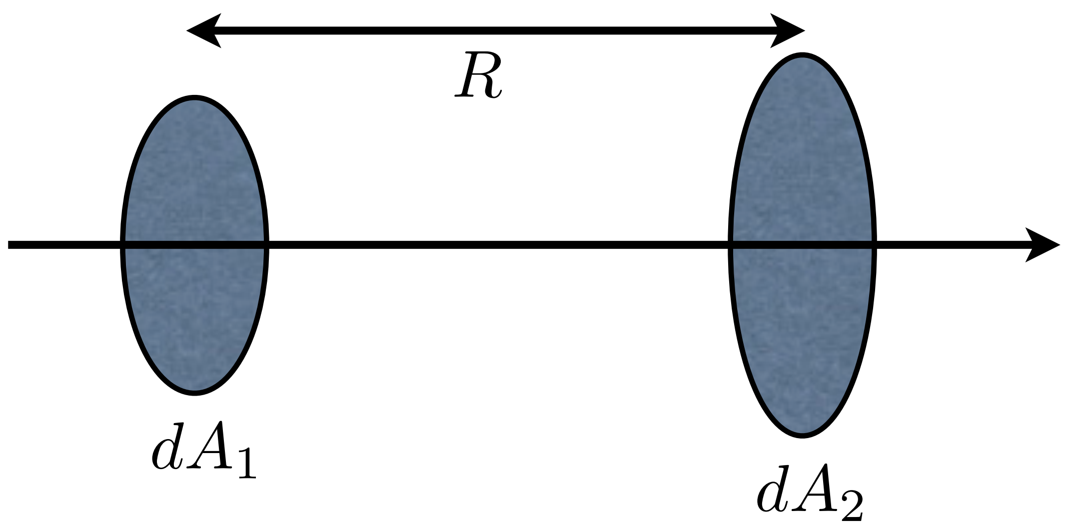
Consider a bundle of rays and any two points along the rays. Construct areas dA_1 and dA_2 normal to the rays at these points.

Consider the energy carried by the rays passing through both areas. Because energy is conserved,

$$dE_1 = I_{\nu_1} dA_1 dt d\Omega_1 d\nu_1 = dE_2 = I_{\nu_2} dA_2 dt d\Omega_2 d\nu_2$$

Here, $d\Omega_1$ is the solid angle subtended by dA_2 at dA_1 and so forth.

$$d\Omega_1 = dA_2/R^2$$
$$d\Omega_2 = dA_1/R^2 \rightarrow I_{\nu_1} = I_{\nu_2}$$
$$d\nu_1 = d\nu_2$$



radiative transfer equation in free space:

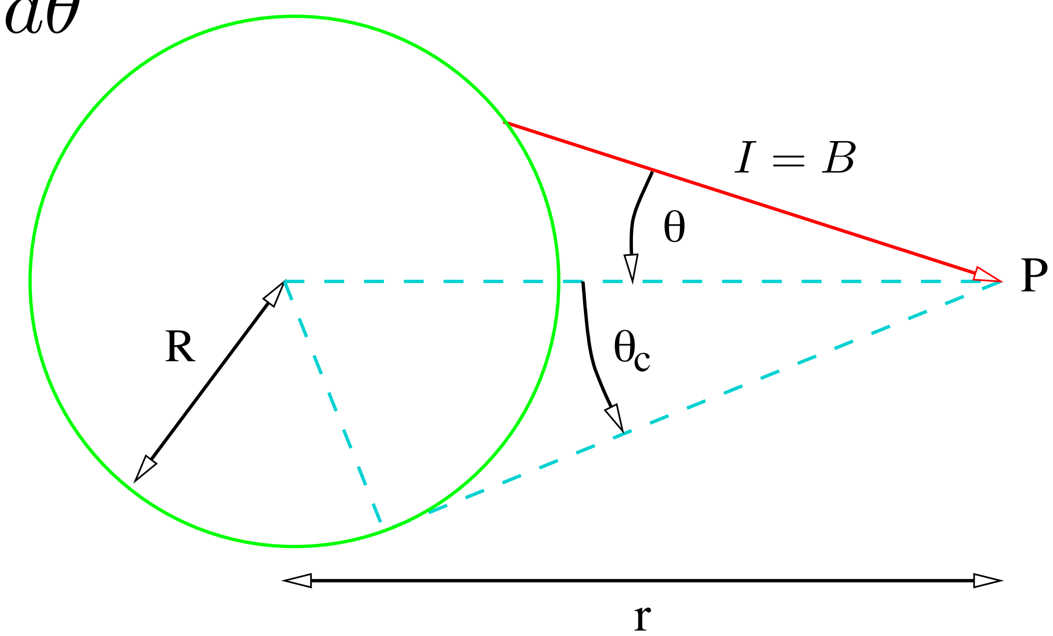
$$\frac{dI_\nu}{ds} = 0$$

Inverse Square Law for a Uniformly Bright Sphere

Let's calculate the flux at P from a sphere of uniform brightness B

$$F = \int I \cos\theta d\Omega = B \int_0^{\pi} d\phi \int_0^{\theta_c} \cos\theta \sin\theta d\theta$$
$$= \pi B(1 - \cos^2\theta_c) = \pi B \sin^2\theta_c$$

$$\sin\theta_c = \frac{R}{r} \rightarrow F = \pi B \left(\frac{R}{r}\right)^2$$



Therefore, there is no conflict between the constancy of intensity and the inverse square law.

Note

- The flux at a surface of uniform brightness B is $F = \pi B$.
- For stellar atmosphere, the astrophysical flux is defined by F/π.

(Specific) Energy Density

Consider a bundle of rays passing through a volume element dV in a direction Ω.

Then, the energy density per unit solid angle is defined by

$$dE = u_\nu(\Omega) dV d\Omega d\nu$$

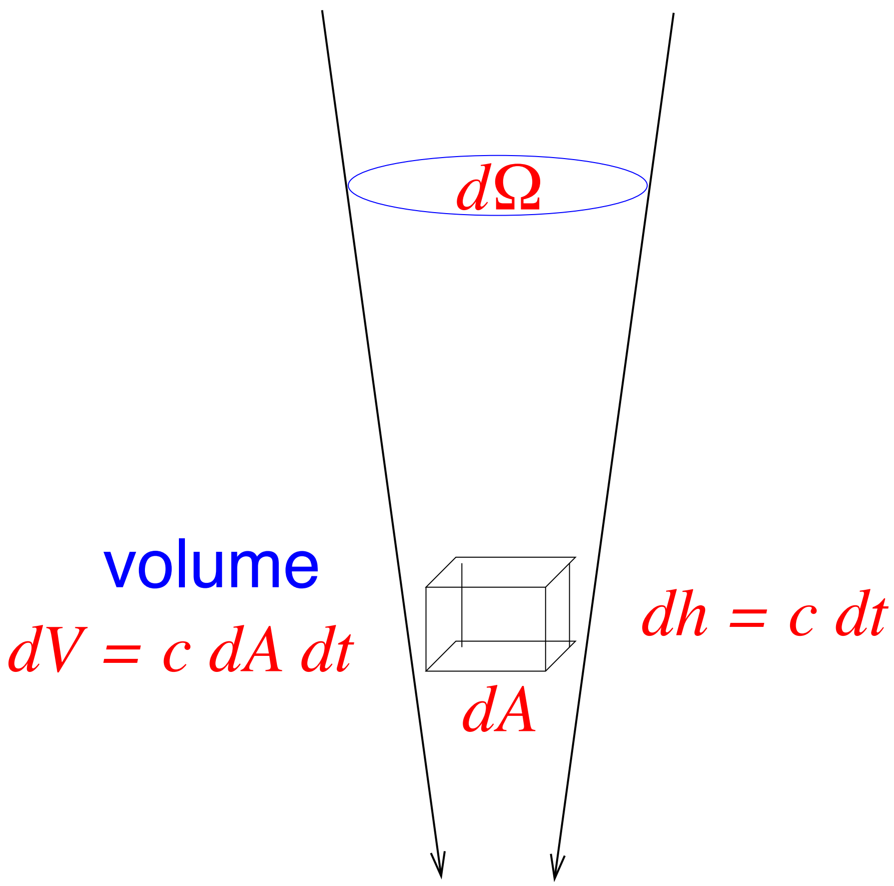
Since radiation travels at velocity c, $dV = dA(cdt)$

the definition of the intensity

$$dE = I_\nu dA dt d\Omega d\nu$$

Therefore,

$$u_\nu(\Omega) = I_\nu(\Omega)/c$$



Energy Density and Mean Intensity

Integrating over all solid angle, we obtain

$$u_\nu = \int u_\nu(\Omega)d\Omega = \frac{1}{c}\int I_\nu d\Omega$$

Mean intensity is defined by

$$J_\nu = \frac{1}{4\pi}\int I_\nu d\Omega$$

Then, the energy density is

$$u_\nu = \frac{4\pi}{c}J_\nu$$

Total energy density is obtained by integrating over all frequencies.

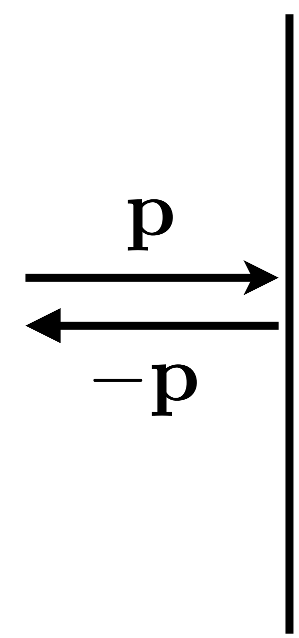
$$u = \int u_\nu d\nu = \frac{4\pi}{c}\int J_\nu d\nu$$

Radiation Pressure

Consider a reflecting enclosure containing an isotropic radiation field.

Each photon transfers twice its normal component of momentum on reflection. Thus, we have

$$
\begin{aligned}
p_\nu &= \frac{2}{c}\int I_\nu \cos^2\theta d\Omega \\
&= \frac{2}{c}J_\nu \int \cos^2\theta d\Omega \\
&= \frac{4\pi}{c}J_\nu \int_0^1 \mu^2 d\mu \\
&= \frac{1}{3}u_\nu
\end{aligned}
$$



$$\Delta p = 2p$$

The angular integration yields $p = \frac{1}{3}u$

Radiative Transfer Equation

As a ray passes through matter, energy may be added, subtracted, or scattered from it by emission, absorption, or scattering.

The intensity will not in general remain constant.

We need to derive the radiative transfer equation.

Emission coefficient and Emissivity

- (monochromatic) spontaneous emission coefficient j_ν = the energy emitted per unit time per unit solid angle and per unit volume

$$dE = j_\nu dV d\Omega dt d\nu \quad (j_\nu : \mathrm{erg\ cm^{-3}\ s^{-1}\ sr^{-1}\ Hz^{-1}})$$

- (angle integrated) emissivity ϵ_ν = the energy emitted spontaneously per unit frequency per unit time per unit mass. For isotropic emission,

$$dE = \epsilon_\nu \rho dV dt d\nu \frac{d\Omega}{4\pi} \quad (\epsilon_\nu : \mathrm{erg\ g^{-1}\ s^{-1}\ Hz^{-1}})$$

- Then, we obtain $\quad j_\nu = \dfrac{\epsilon_\nu \rho}{4\pi} \quad \text{or} \quad \displaystyle\int j_\nu d\Omega = \epsilon_\nu \rho$

- In going a distance ds, a beam of cross section dA travels through a volume $dV = dAds$. Thus the intensity added to the beam is by spontaneous emission is:

$$dI_\nu = j_\nu ds$$

Absorption Coefficient

Consider the medium with particle number density $n\ (\mathrm{cm}^{-3})$, each having effective absorbing area (cross section) $\sigma_\nu\ (\mathrm{cm}^2)$

- number of absorbers = $n dA ds$
- total absorbing area = $n\sigma_\nu dA ds$

energy taken out of beam

$$-dI_\nu dA d\Omega dt d\nu = I_\nu n \sigma_\nu dA ds d\Omega dt d\nu$$

$$\rightarrow dI_\nu = -n\sigma_\nu I_\nu ds = -\alpha_\nu I_\nu ds$$

Absorption coefficient $\alpha_\nu\ (\mathrm{cm}^{-1})$ is defined by

$$\begin{aligned} \alpha_\nu &= n\sigma_\nu \\ &= \rho\kappa_\nu \end{aligned}$$

where $\rho\ (\mathrm{g\ cm}^{-3})$ is the mass density and $\kappa_\nu\ (\mathrm{cm}^2\ \mathrm{g}^{-1})$ is called the mass absorption coefficient or the opacity coefficient.

The Radiative Transfer Equation

Without scattering term,

$$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu$$

Including scattering term, we obtain an integrodifferential equation.

$$\mathbf{\Omega} \cdot \nabla I_\nu = -\alpha_\nu^{\mathrm{ext}} I_\nu + j_\nu + \alpha_\nu^{\mathrm{sca}} \int \phi_\nu(\mathbf{\Omega}, \mathbf{\Omega}') I_\nu(\mathbf{\Omega}') d\Omega'$$

- scattering coefficient $\alpha_\nu^{\mathrm{sca}}\ (\mathrm{cm}^{-1})$
- scattering phase function $\int \phi_\nu(\mathbf{\Omega}, \mathbf{\Omega}') d\Omega = 1$
- for isotropic scattering $\phi_\nu(\mathbf{\Omega}, \mathbf{\Omega}') = \frac{1}{4\pi}$

Stimulated emission:

- We consider “absorption” to include both “true absorption” and stimulated emission, because both are proportional to the intensity of the incoming beam (unlike spontaneous emission).

Emission Only & Absorption Only

For pure emission, $\alpha_\nu = 0$

$$\frac{dI_\nu}{ds} = j_\nu \quad \rightarrow \quad I_\nu(s) = I_\nu(s_0) + \int_{s_0}^{s} j_\nu(s')ds'$$

- The brightness increase is equal to the emission coefficient integrated along the line of sight.

For pure absorption, $j_\nu = 0$

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu \quad \rightarrow \quad I_\nu(s) = I_\nu(s_0)\exp\left[-\int_{s_0}^{s} \alpha_\nu(s')ds'\right]$$

- The brightness decreases along the ray by the exponential of the absorption coefficient integrated along the line of sight.

Optical Depth & Source Function

Optical depth:

$$\tau_\nu(s) = \int_{s_0}^{s} \alpha_\nu(s')ds' = \int_{s_0}^{s} n(s')\sigma_\nu ds' = \int_{s_0}^{s} \rho(s')\kappa_\nu ds'$$

- Then, for pure absorption, $I_\nu(s) = I_\nu(s_0)e^{-\tau_\nu}$
- A medium is said to be optically thick if $\tau_\nu > 1$
- A medium is said to be optically thin if $\tau_\nu < 1$

Source function: $S_\nu = \dfrac{j_\nu}{\alpha_\nu}$

- The radiative transfer equation can now be written

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

Mean Free Path

- From the exponential absorption law, the probability of a photon absorbed between optical depths τ_ν and $\tau_\nu + d\tau_\nu$:

$$|dI_\nu| = \left|\frac{dI_\nu}{d\tau_\nu}\right| d\tau_\nu \quad \& \quad |dI_\nu| \propto P(\tau_\nu) d\tau_\nu \quad \rightarrow \quad P(\tau_\nu) = e^{-\tau_\nu}$$

- The mean optical depth traveled is thus equal to unity:

$$< \tau_\nu >= \int_0^\infty \tau_\nu P(\tau_\nu) d\tau_\nu = \int_0^\infty \tau_\nu e^{-\tau_\nu} d\tau_\nu = 1$$

- The mean free path is defined as the average distance a photon can travel through an absorbing material without being absorbed. In a homogeneous medium, the mean free path is determined by

$$< \tau_\nu >= \alpha_\nu l_\nu = 1 \quad \rightarrow \quad l_\nu = \frac{1}{\alpha_\nu} = \frac{1}{n\sigma_\nu}$$

- A local mean path at a point in an inhomogeneous material can be also defined.

Formal Solution

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$$

$$e^{\tau_\nu}\frac{dI_\nu}{d\tau_\nu} + e^{\tau_\nu} I_\nu = e^{\tau_\nu} S_\nu$$

$$\frac{d}{d\tau_\nu}\left(e^{\tau_\nu} I_\nu\right) = e^{\tau}_{\nu} S_\nu$$

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)} S_\nu(\tau'_\nu) d\tau'_\nu$$

The solution is easily interpreted as the sum of two terms:

- the initial intensity diminished by absorption
- the integrated source diminished by absorption.

Relaxation

For a constant source function, the solution becomes

$$\begin{aligned} I_\nu(\tau_\nu) &= I_\nu(0)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu}) \\ &= S_\nu + e^{-\tau_\nu}\left(I_\nu(0) - S_\nu\right) \end{aligned}$$

"Relaxation"

- $I_\nu > S_\nu \quad \rightarrow \quad \dfrac{dI_\nu}{d\tau_\nu} < 0, \quad \text{then } I_\nu \text{ tends to decrease along the ray}$

  $I_\nu < S_\nu \quad \rightarrow \quad \dfrac{dI_\nu}{d\tau_\nu} > 0, \quad \text{then } I_\nu \text{ tends to increase along the ray}$

- The source function is the quantity that the specific intensity tries to approach, and does approach if given sufficient optical depth.

$$\text{As} \quad \tau_\nu \rightarrow \infty, \quad I_\nu \rightarrow S_\nu$$

Radiation Force

- radiation flux vector in direction $\mathbf{n}$:

$$\mathbf{F}_\nu = \int I_\nu \mathbf{n} d\Omega$$

- the vector momentum per unit area per unit time per unit path length absorbed by the medium is

$$\mathsf{F} = \frac{1}{c}\int \alpha_\nu \mathbf{F}_\nu d\nu \quad \leftarrow \quad n\sigma_\nu dA ds \frac{\mathbf{F}_\nu}{c}$$

- The is the force per unit volume imparted onto the medium by the radiation field. The force per unit mass of material is given by

$$\mathsf{f} = \frac{\mathsf{F}}{\rho} = \frac{1}{c}\int \kappa_\nu \mathbf{F}_\nu d\nu$$

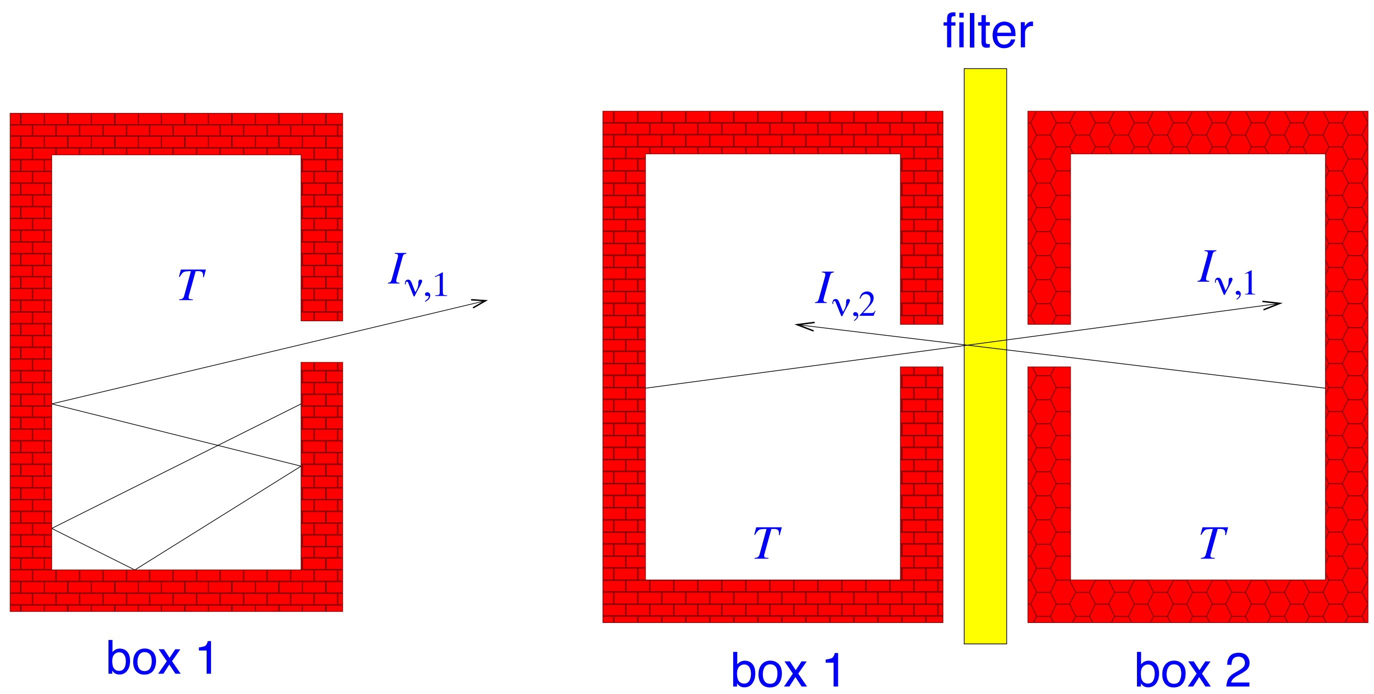
- Home work: derive the Eddington luminosity (problem 1.4)

Thermal equilibrium

- Equilibrium means a state of balance.
- Thermal equilibrium refers to steady states of temperature, which may be spatial or temporal.
- In a state of (complete) thermodynamic equilibrium, no net flows of matter or of energy, no phase changes, and no unbalanced potentials (or driving forces), within the system.
- When the material is in thermodynamic equilibrium, and only the radiation field is allowed to depart from its TE, we refer to the state of the system as being in local thermodynamic equilibrium (LTE).
- A blackbody is an idealized physical body that absorbs all incident radiation regardless of frequency or angle of incidence (i.e., perfect absorber).
- A blackbody in thermal equilibrium emits the blackbody radiation, which is itself in thermal equilibrium.
- Thermal radiation is radiation emitted by “matter” in thermal equilibrium.

Universal function

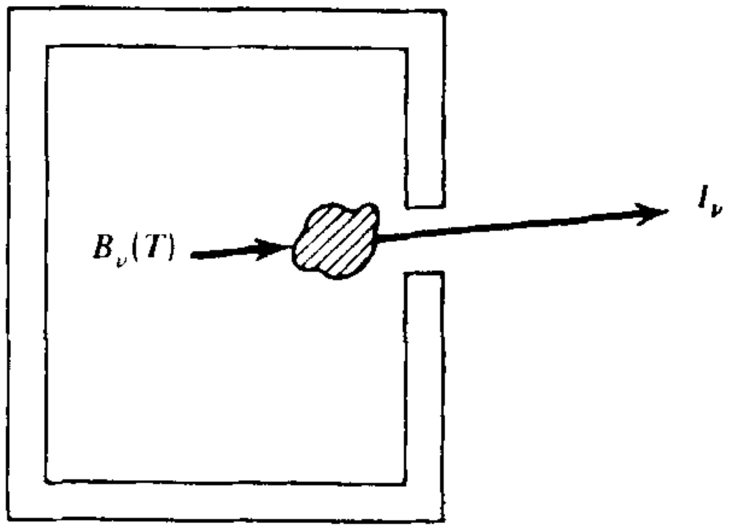
- In equilibrium, radiation field in box doesn't change.



- Now, consider another enclosure (box 2), also at the same temperature, but made of different material or shape. If their escaping intensities are different, energy will flow spontaneously between the two boxes. This violets the second law of thermodynamics. Therefore, the escaping intensity should be a universal function of T and should be isotropic. The universal function is called the Planck function $B_\nu(T)$.

Kirchhoff's Law

- Consider an element of some thermally emitting material at temperature T.
- Put this into a blackbody enclosure at the same temperature.
- Let the source function of the material be S_ν .



$$\text{if } S_\nu > B_\nu \quad \rightarrow \quad I_\nu > B_\nu$$
$$\text{if } S_\nu < B_\nu \quad \rightarrow \quad I_\nu < B_\nu$$
$$\therefore \ S_\nu = B_\nu$$

- But, the presence of the material cannot alter the radiation, since the new configuration is also a blackbody enclosure at T.
- Kirchhoff's Law: in LTE, the ratio of the emission coefficient to the absorption coefficient is a function of temperature and frequency only.

$$j_\nu = \alpha_\nu B_\nu(T) \ \rightarrow \ \text{Kirchhoff's Law}$$

$$\text{Note} : j_\nu = B_\nu(T) \ \text{ if } \ \alpha_\nu = 1 \ (\text{perfect absorber, i.e., blackbody})$$

Implications of Kirchhoff's Law

- A good absorber is a good emitter, and a poor absorber is a poor emitter. A good reflector must be a poor absorber, and thus a poor emitter.
- It is not possible to thermally radiate more energy than a blackbody, at equilibrium.

$$j_\nu \leq B_\nu(T)$$

- The radiative transfer equation in LTE:

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + B_\nu(T)$$

- Note:

blackbody radiation means $I_\nu = B_\nu(T)$

thermal radiation means $S_\nu = B_\nu(T)$

Thermal radiation becomes blackbody radiation only for optically thick media.

Thermodynamics

- First Law of Thermodynamics: heat is energy in transit.

$$dQ = dU + pdV$$

where Q is heat and U is total energy.

- Second Law of Thermodynamics: heat is entropy.

$$dS = \frac{dQ}{T}$$

where S is entropy.

See “Fundamentals of Statistical and Thermal Physics” (Federick Reif)

Thermodynamics of Blackbody Radiation

$$
\begin{aligned}
dS = \frac{dU}{T} + p\frac{dV}{T} &= \frac{1}{T}d(uV) + \frac{1}{3}\frac{u}{T}dV \\
&= \frac{V}{T}\frac{du}{dT}dT + \frac{u}{T}dV + \frac{1}{3}\frac{u}{T}dV \\
&= \frac{V}{T}\frac{du}{dT}dT + \frac{4}{3}\frac{u}{T}dV
\end{aligned}
$$

$$\rightarrow \quad \frac{\partial S}{\partial T} = \frac{V}{T}\frac{du}{dT} \quad \text{and} \quad \frac{\partial S}{\partial V} = \frac{4u}{3T}$$

$$\rightarrow \quad \frac{\partial^2 S}{\partial T \partial V} = \frac{1}{T}\frac{u}{T} = -\frac{4u}{3T^2} + \frac{4}{3T}\frac{du}{dT}$$

$$\therefore \frac{du}{dT} = \frac{4u}{T} \quad \text{or} \quad \frac{du}{u} = 4\frac{dT}{T}$$

- Stefan-Boltzmann law:

$$\mathrm{Stenfan-Boltzmann\ law:}\quad u(T) = aT^4 \;\leftarrow\; \log u = 4\log T + \log a$$

$$u(T) = \left(\frac{T}{3400\ K}\right)^4 \ \mathrm{erg\ cm^{-3}}$$

- total energy density:

$$u = \frac{4\pi}{c}\int B_\nu(T)d\nu = \frac{4\pi}{c}B(T)$$

- the integrated Planck function

$$B(T) = \int B_\nu(T)d\nu = \frac{ac}{4\pi}T^4 = \frac{\sigma}{\pi}T^4$$

- emergent flux (another form of the Stefan-Boltzmann law)

$$F = \int F_\nu d\nu = \pi\int B_\nu d\nu = \pi B(T)$$
$$= \sigma T^4$$

Entropy of Blackbody Radiation

- Entropy:

$$dS = \frac{V}{T}4aT^3dT + \frac{4a}{3}T^3dV \quad \rightarrow \quad S = \frac{4}{3}aT^3V$$

- Entropy density:

$$s = S/V = \frac{4}{3}T^3$$

- The law of adiabatic expansion for blackbody radiation:

$$T_{\rm ad} \propto V^{-1/3}$$

$$p_{\rm ad} \propto T_{\rm ad}^4 \propto V^{-4/3}$$

Thus, we have the adiabatic index for blackbody radiation:

$$\gamma = \frac{4}{3} \quad \leftarrow \quad pV^\gamma = \text{constant}$$

The Planck Spectrum

Density of photon state:

- Consider a photon propagating in direction $\mathbf{n}$ inside a box with dimensions L_x, L_y, L_z in x, y, z directions.
- wave vector: $\mathbf{k} = \frac{2\pi}{\lambda}\mathbf{n} = \frac{2\pi\nu}{c}\mathbf{n}$
- If each dimension of the box is much longer than a wavelength, the photon can be represented by standing wave in the box.
- number of nodes in each direction: $n_x = k_x L_x / 2\pi$
- number of node changes in a wave number interval (if $n_i \gg 1$):

$$\Delta n_x = \frac{L_x \Delta k_x}{2\pi}$$

- number of stats in 3D wave vector element $\Delta k_x \Delta k_y \Delta k_z = d^3k$:

$$\Delta N = \Delta n_x \Delta n_y \Delta n_z = 2\frac{L_x L_y L_z d^3k}{(2\pi)^3} = 2\frac{V d^3k}{(2\pi)^3}$$

two independent polarizations

- the density of states (number of states per solid angle per volume per frequency):

$$d^3k = k^2 dk d\Omega = \frac{(2\pi)^3 \nu^2 d\nu d\Omega}{c^3}$$

$$\rightarrow \quad \rho_s = \frac{dN}{V d\nu d\Omega} = \frac{2\nu^2}{c^3}$$

Average energy of each state:

- Each state may contain n photons of energy $h\nu$. The energy of the state is $E_n = nh\nu$.
- The probability of a state of energy E_n is proportional to $e^{-\beta E_n}$, where $\beta = (k_\mathrm{B} T)^{-1}$ and k_B is the Boltzmann's constant.
- Therefore, the average energy is:

$$\langle E \rangle = \frac{\Sigma_{n=0}^{\infty} E_n e^{-\beta E_n}}{\Sigma_{n=0}^{\infty} e^{-\beta E_n}} = -\frac{\partial}{\partial \beta} \ln \left(\Sigma_{n=0}^{\infty} e^{-\beta E_n} \right)$$

$$\Sigma_{n=0}^{\infty} e^{-\beta E_n} = \Sigma_{n=0}^{\infty} \left(e^{-\beta h\nu}\right)^n = \left(1 - e^{-\beta h\nu}\right)^{-1}$$

$$\langle E \rangle = \frac{h\nu e^{-\beta h\nu}}{1 - e^{-\beta h\nu}} = \frac{h\nu}{\exp(h\nu/k_{\rm B}T) - 1}$$

Average number of photons (occupation number):

$$n_\nu = \langle E \rangle / h\nu = \frac{1}{\exp(h\nu/k_{\rm B}T) - 1} \rightarrow \text{Bose} - \text{Einstein statistics}$$

Energy density:

$$u_\nu(\Omega) = \rho_s \langle E \rangle = \frac{2h\nu^3/c^3}{\exp(h\nu/k_{\rm B}T) - 1}$$

Planck Law:

$$B_\nu = \frac{2h\nu^3/c^2}{\exp(h\nu/k_{\rm B}T) - 1} \text{ or } B_\lambda = \frac{2hc^2/\lambda^5}{\exp(hc/\lambda k_{\rm B}T) - 1}$$

Stefan-Boltzmann constant & Riemann zeta function

$$\text{Bose integral}: \quad I_n = \int_0^\infty dx \frac{x^n}{e^x - 1} = \int_0^\infty dx x^n \sum_{i=0}^{\infty} e^{-(i+1)x}$$

$$= \sum_{i=0}^{\infty} \frac{1}{(i+1)^{n+1}} \int_0^\infty dy y^n e^{-y} \quad (y \equiv (i+1)x)$$

$$= \zeta(n+1)\Gamma(n+1)$$

$$\int_0^\infty B_\nu(T) d\nu = (2h/c^2)(k_\mathrm{B}T/h)^4 \int_0^\infty \frac{x^3 dx}{e^x - 1}$$

$$= \frac{2k_\mathrm{B}^4 T^4}{c^2 h^3} \zeta(4)\Gamma(4) = \frac{2k_\mathrm{B}^4 T^4}{c^2 h^3} \frac{\pi^4}{90} 6$$

$$= \frac{2\pi^4 k_\mathrm{B}^4}{15 c^2 h^3} T^4$$

$$\therefore \quad \sigma = \frac{2\pi^5 k_\mathrm{B}^5}{15 c^2 h^3} = 5.67 \times 10^{-5} \text{ erg cm}^{-2} \text{ deg}^{-4} \text{ s}^{-1}$$

$$a = \frac{8\pi^5 k_\mathrm{B}^4}{15 c^3 h^3} = 7.56 \times 10^{-15} \text{ erg cm}^{-3} \text{ deg}^{-4}$$

Rayleigh-Jeans Law & Wien Law

Rayleigh-Jeans Law

$$h\nu \ll k_{\rm B}T \ (\nu \ll 2\times 10^{10}{\rm Hz}(T/1{\rm K})) \quad \rightarrow \quad I_\nu^{RJ}(T) = \frac{2\nu^2}{c^2}k_{\rm B}T$$

- Originally derived by assuming the classical equipartition energy

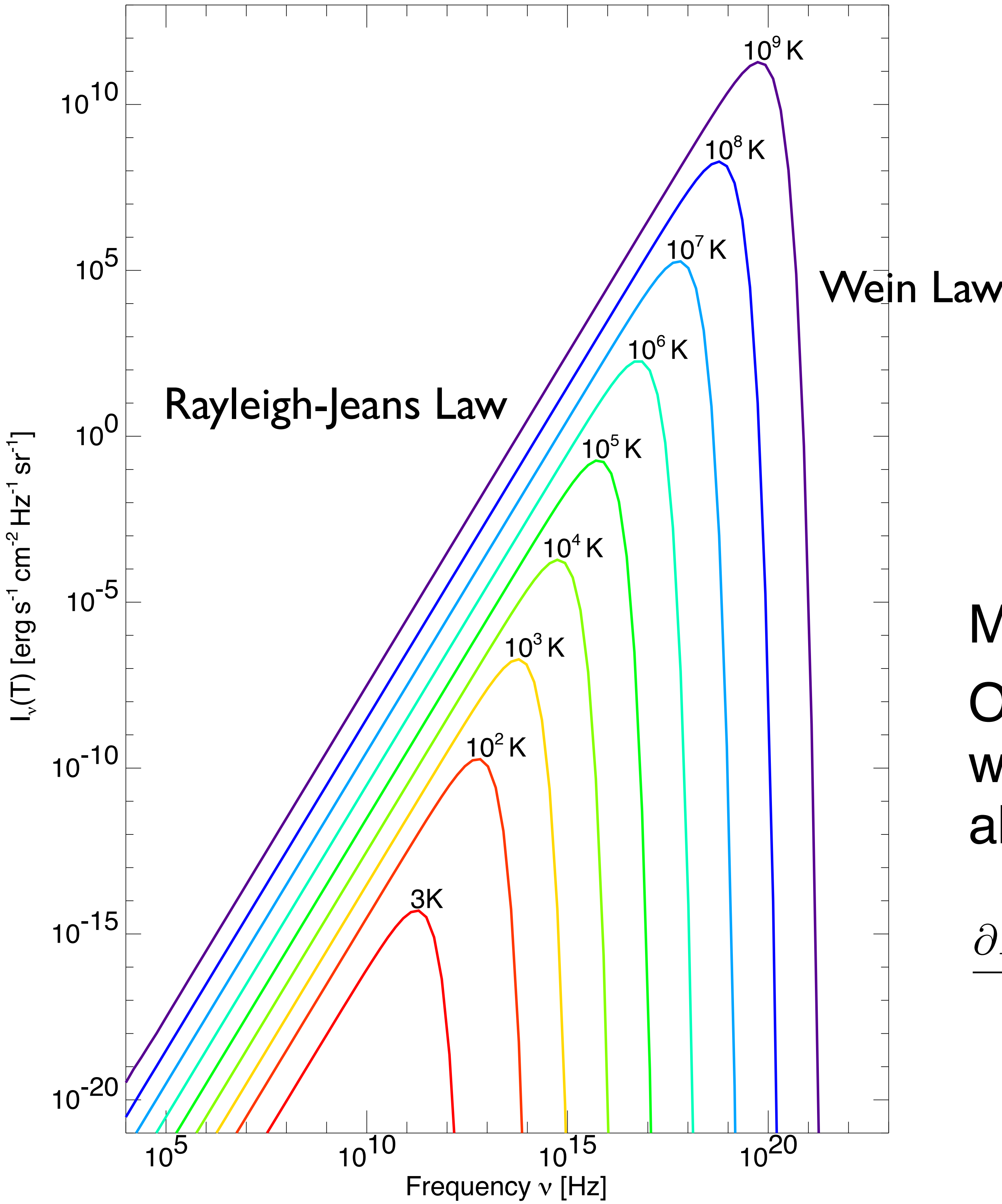
$$\langle E \rangle = 2\times(1/2)k_{\rm B}T$$

- ultraviolet catastrophe: if the equation is applied to all frequencies, the total amount of energy would diverge. $\int \nu^2 d\nu \rightarrow \infty$

Wien Law

$$h\nu \gg k_{\rm B}T \quad \rightarrow \quad I_\nu^{W}(T) = \frac{2h\nu^3}{c^2}\exp\left(-\frac{h\nu}{k_{\rm B}T}\right)$$

Monotonicity with Temperature



Monotonicity:

Of two blackbody curves, the one with higher temperature lies entirely above the other.

$$\frac{\partial B_\nu(T)}{\partial T} = \frac{2h^2\nu^4}{c^2 k_B T^2} \frac{\exp(h\nu/k_T)}{[\exp(h\nu/k_B T) - 1]^2} > 0$$

Wien Displacement Law

Frequency at which the peak occurs:

$$\frac{\partial B_\nu}{\partial \nu}|_{\nu=\nu_{\rm max}} = 0 \quad \rightarrow \quad x = 3(1-e^{-x}), \text{ where } x = h\nu_{\rm max}/k_{\rm B}T$$

$$h\nu_{\rm max} = 2.82 k_{\rm B}T \quad \text{or} \quad \frac{\nu_{\rm max}}{T} = 5.88 \times 10^{10} \text{ Hz deg}^{-1}$$

Wavelength at which the peak occurs:

$$\frac{\partial B_\lambda}{\partial \lambda}|_{\lambda=\lambda_{\rm max}} = 0 \quad \rightarrow \quad y = 5(1-e^{-y}), \text{ where } y = hc/(\lambda_{\rm max}k_{\rm B}T)$$

$$y = 4.97 \quad \text{and} \quad \lambda_{\rm max}T = 0.290 \text{ cm deg}$$

Be aware $\nu_{\rm max} \neq c/\lambda_{\rm max}$

Characteristic Temperatures

Brightness Temperature:

$$I_\nu = B_\nu(T_b)$$

- The definition is used especially in radio astronomy, where the RJ law is usually applicable. In the RJ limit,

$$T_b = \frac{c^2}{2\nu^2 k_{\rm B}} I_\nu$$

- radiative transfer equation in the RJ limit:

$$\frac{dT_b}{d\tau_\nu} = -T_b + T \quad (T = \text{the temperature of the material})$$

$$T_b = T_b(0)e^{-\tau_\nu} + T(1 - e^{-\tau_\nu}) \quad \text{if } T \text{ is constant.}$$

- In the Wien region, the concept is not so useful.

Characteristic Temperatures

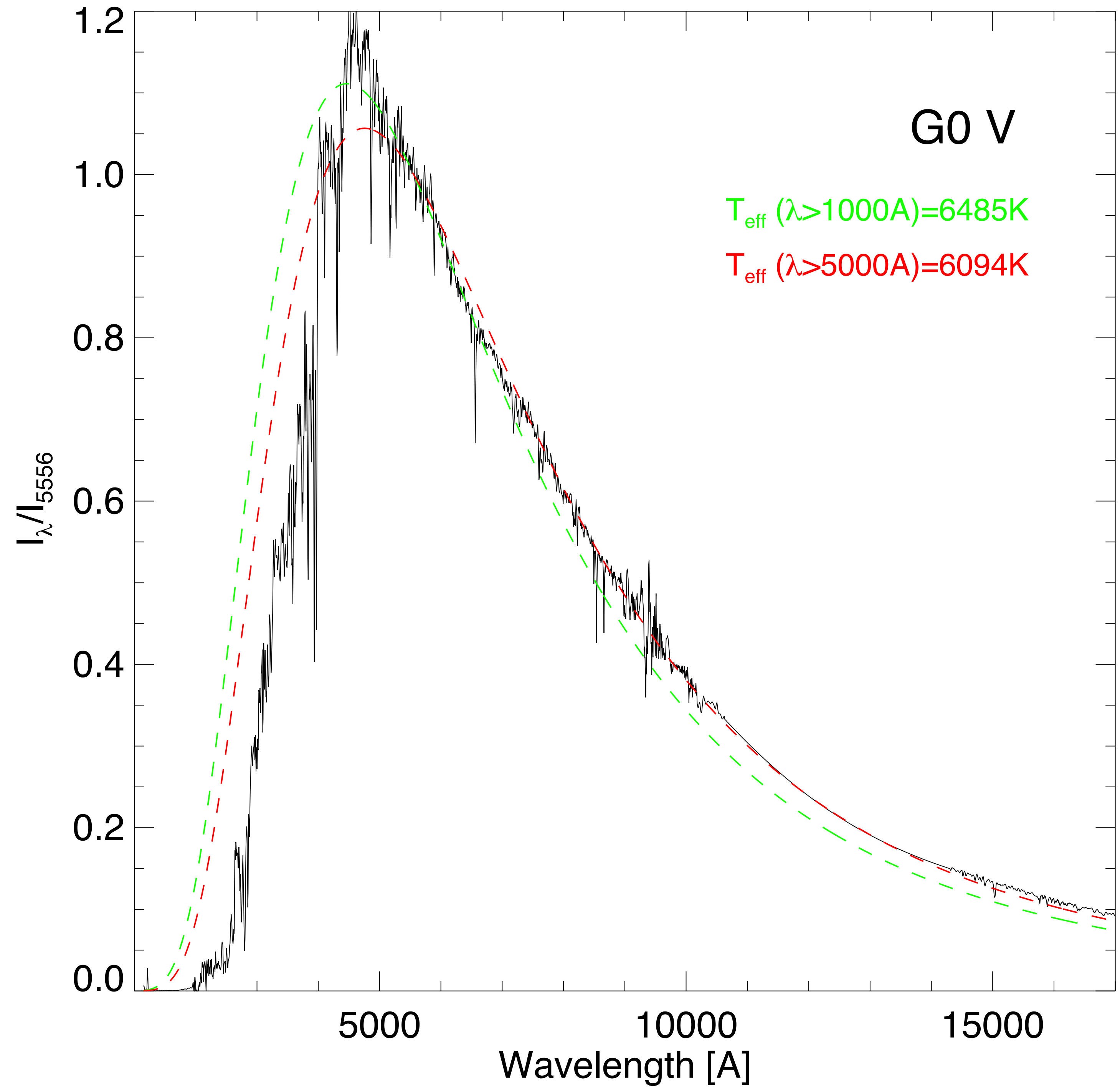
Color Temperature:

- By fitting the spectrum to a blackbody curve without regarding to vertical scale (absolute intensity scale), a color temperature T_c is obtained.
- The color temperature correctly gives the temperature of a blackbody source of unknown absolute scale.

Effective Temperature:

- The effective temperature of a source is obtained by equating the actual flux F to the flux of a blackbody at temperature $T_{\rm eff}$.

$$F = \int \cos\theta I_\nu d\nu d\Omega = \sigma T_{\rm eff}^4$$



G0V spectrum (Pickles 1998, PASP, 110, 863)